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The design of high speed digital systems requires careful effort to minimize the propagation delay through combinational logic networks. For technological reasons these networks are often most desirably constructed from either NAND or NOR gates with various fan-in limits. It is simpler for the designer to write Boolean expressions which define the desired network behavior in terms of the general class of binary Boolean operators (henceforth referred to as B2) instead of multiple input NAND or NOR operators. → next page

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The main objective of this paper is to provide a methodology for convenient design of combinational networks. The techniques provided here are not merely existence proofs but are constructive and could, for example, be implemented in the form of computer programs to convert expressions with very large numbers of literals to networks meeting the indicated bounds, leaving only small (eg. 3 to 5 literals) networks to be constructed by hand or through table search.

All bounds and constructions in this paper are based upon the size of the input expression where for any expression E the size of E is represented by |E| and is equal to the number of literals in E. The techniques presented treat each literal as distinct (eg. AB+AC has 4 literals). Thus, these techniques are most useful when the size of the expression is large and the number of repetitions of variable names is relatively small.

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DEPTH ORIENTED DESIGN FOR NAND NETWORKS

by

James Paul Rutledge

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DEPTH ORIENTED DESIGN FOR NAND NETWORKS

BY

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THESIS

Submitted in partial fulfillment of the requirements
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Thesis Adviser: Professor F. P. Preparata

Urbana, Illinois

DEPTH ORIENTED DESIGN FOR NAND NETWORKS

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Coordinated Science Laboratory and
Department of Electrical Engineering
University of Illinois at Urbana-Champaign, 1978

The design of high speed digital systems requires careful effort to minimize the propagation delay through combinational logic networks. For technological reasons these networks are often most desirably constructed from either NAND or NOR gates with various fan-in limits. It is simpler for the designer to write Boolean expressions which define the desired network behavior in terms of the general class of binary Boolean operators (henceforth referred to as B2) instead of multiple input NAND or NOR operators.

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literals). Thus, these techniques are most useful when the size of the expression is large and the number of repetitions of variable names is relatively small.

All logarithms are taken to base 2.

Input literals are assumed to be available in both complemented and uncomplemented form (2-rail inputs) and to have unlimited fan-out available.

An outline of the key contents of this paper is provided as follows:

1. Introduction
   - A. Conversion of NAND network algorithms for NOR network construction.
2. Formula conversion for depth reduction.
   - A. McColl's B2 to 2-input NAND conversion.
     Depth Bound: $2.88 \log |E| + O(1)$
   - B. Extension of Pratt's work to B2 to 2-input NAND conversion.
     Depth Bound: $2.88 \log |E| + O(1)$
   - C. Conjecture that B2 to 2-input NAND conversion cannot be done with a better upper bound than $2.88 \log |E| + O(1)$.
   - D. B2 to 3-input NAND conversion.
     Depth Bound: $2 \log |E| + O(1)$
   - E. B2 to 4-input NAND conversion.
     Depth Bound: $1.44 \log |E| + O(1)$

3. Equipment complexity corresponding to restructured formulae.

A. 2-input NAND.

(1) Method 1.

Equipment Bound: $O(|E|^{1.832})$

(2) Method 2.

Equipment Bound: $O(|E|^{1.44})$

B. 3-input NAND.

Equipment Bound: $O(|E|^{2.216})$

C. 4-input NAND.

Equipment Bound: $O(|E|^{1.832})$

4. Formula conversion for depth reduction with linear equipment.

A. B2 to 2-input NAND.

Depth Bound: $4 \log |E| + O(1)$

B. B2 to 3-input NAND

Depth Bound: $2.465 \log |E| + O(1)$

C. B2 to 4-input NAND.

Depth Bound: $2 \log |E| + O(1)$

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I wish to thank the United States Air Force for providing me the opportunity for graduate study. Thanks are also due my advisor, Dr. F.P. Preparata, for his time, patience, and encouragement. A special debt is owed to Dr. Preparata and Dr. D.E. Muller for their pioneering work in the restructuring of arithmetic and Boolean expressions, in which a previously little understood art of restructuring was brought to sophisticated levels and extended to multiple input gates. I have made extensive use of their techniques in this work. The time and effort expended by Dr. Muller and Dr. D.V. Sarwate as members of my committee is also much appreciated. Finally, to my wife, Brenda Kay, who has endured much during this time, yet has both kept her good humor and done yoeman work to keep our home shipshape, I wish to express my appreciation.

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CHAPTER 1

INTRODUCTION

The design of high speed digital systems requires careful effort to minimize the propagation delay through combinational logic networks. For technological reasons these networks are often most desirably constructed from either NAND or NOR gates with various fan-in limits. It is simpler for the designer to write Boolean expressions which define the desired network behavior in terms of the general class of binary Boolean operators (henceforth referred to as B2) instead of multiple input NAND or NOR operators.

The problem which thus presents itself is the construction of networks of either NAND or NOR gates (with various fan-in limits) which realize the function represented by any given Boolean expression using operators in B2. Further, these networks must provide "good" asymptotic behavior with respect to depth as a function of the "size" of the expression.

The work described in this paper is based primarily upon the papers by Preparata and Muller [10]; and Preparata, Muller, and Barak [9]. In those two papers restructuring techniques for Boolean expressions were brought to a greatly more sophisticated level than had been obtained previously, allowing powerful depth limiting

All logarithms are taken to base 2.

Input literals are assumed to be available in both complemented and uncomplemented form (2-rail inputs) and to have unlimited fan-out available.

1.2 OUTLINE OF CONTENTS

An outline of the key contents of this paper is provided as follows:

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   - E. B2 to 4-input NAND conversion.
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3. Equipment complexity corresponding to restructured

Formula conversion is the process of translating a given formula using binary Boolean connectives from one basis into a formula which is equivalent to the original formula (in the sense that the two Boolean functions represented by the two formulas are identical) and uses binary Boolean connectives from another basis.

For example, the formula shown in figure 1 is expressed in the basis B2 (with symbol ⊕ meaning EXCLUSIVE-OR), but could be converted to the basis NAND as shown in figure 2 (with two rail inputs assumed to be available and o's used to represent NAND gates).

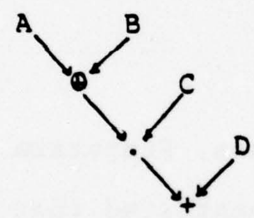

FIGURE 1 ORIGINAL FORMULA

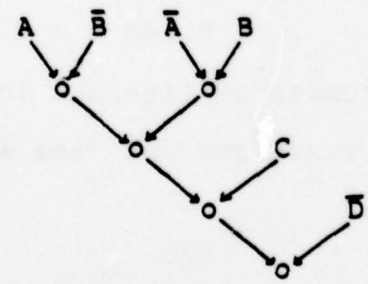

FIGURE 2 CONVERTED FORMULA

techniques (also depth limiting expansions giving linear equipment growth) for network implementations using gates with various fan-in limits. Those two papers were directed toward the restructuring of input Boolean expressions described using the binary unate (U2) operators. (U2 is B2 without EXCLUSIVE-OR and EQUIVALENCE.)

The main objective of this paper is to provide a methodology for convenient design of combinational networks. The techniques provided here are not merely existence proofs but are constructive and could, for example, be implemented in the form of computer programs to convert expressions with very large numbers of literals to networks meeting the indicated bounds, leaving only small (eg. 3 to 5 literals) networks to be constructed by hand or through table search.

1.1 PRELIMINARY NOTES

All bounds and constructions in this paper are based upon the size of the input expression where for any expression E the size of E is represented by $|E|$ and is equal to the number of literals in E. The techniques presented treat each literal as distinct (eg. AB+AC has 4 literals). Thus, these techniques are most useful when the size of the expression is large and the number of repetitions of variable names is relatively small.

formulae.

A. 2-input NAND.

(1) Method 1.

Equipment Bound: $O(|E|^{1.832})$

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B. B2 to 3-input NAND

Depth Bound: $2.465 \log |E| + O(1)$

C. B2 to 4-input NAND.

Depth Bound: $2 \log |E| + O(1)$

1.3 FORMULA CONVERSION

A Boolean formula with n inputs is a directed binary tree in which all nodes have out-degree 1 and in-degree 2 (in which case the inputs are ordered) or in-degree 0 (an input node). The in-degree 2 nodes represent binary Boolean operations. The number of input nodes is n.

1.3.1 LEMMA 1

A key development in the techniques for restructuring of formulas was a lemma by Brent, Kuck, and Maruyama [2] in which it was shown that for any binary tree with n leaves and any real number q such that $1<q\leq n$, by following a path from the root such that each edge selected leads toward the immediate subtree with the larger number of leaves, a node will be found such that each of the two immediate subtrees of that node has less than q leaves and the sum of the number of leaves of those two subtrees is greater than or equal to q.

1.3.2 BACKGROUND

Using that lemma, Preparata, Muller, and Barak [9][10] constructively demonstrated that formula conversion from the basis U2 to an AND and OR gate realization (assuming 2 rail inputs) can be done with depth upper bounded by

$$R \log n + S$$

where n is the number of literals in the Boolean formula and R and S are functions of the maximum gate fan-in k, as shown in table 1.

TABLE 1 U2 TO "AND" AND "OR" CONVERSION

k	R	S
2	1.81	0
3	1.38	0
4	1.18	.134
5	1	.415

This incorporation of maximum gate fan-in as a parameter is a very significant result.

McColl [6] has shown that formula conversion from B2 to NAND for a gate fan-in of two can be done with a value of 2.88 for R. He also indicates that formula B2 to B2 restructuring can result in a value of 2.465 for R.

1.3.3 TECHNIQUES

The key technique used in this type of work is "divide and conquer". A formula with n literals is subdivided into formulas with various fractions of n as upper bounds on the number of literals in the portions of the subdivision. These portions are chosen such that they can be combined by a "small" formula to give a formula equivalent to the original formula. This procedure is then applied recursively to the subdivisions, resulting in an overall formula depth of R log n + O(1), where R is determined by the form of the "small" formula and the bounds on the number of literals in the subdivisions.

1.3.4 EXAMPLE OF TECHNIQUES

A brief example (without proof) is given here to provide some insight into the recurrence used. A form of the Fundamental Theorem of Boolean Algebra will be used as the "small" formula, $F=(x \otimes f_0)+(xf_1)$, in the bounding of depth of formulas from B2 to B2, where $\otimes$ is the binary Boolean function given by $a \otimes b = \bar{a}b$.

We use the lemma of Brent et al, with q in the lemma replaced by zn where n is the number of literals in the original formula and z is the only real root of the equation $0 = y^3-2y^2+3y-1$ (z is approximately .43016). The original formula is thus subdivided into 3 parts, A, B, and C, such that the number of literals in each of B and C alone is less than zn, while the number of literals in B and C together is greater than or equal to zn. "A" is the remaining part of the original formula after the removal of B, C, and their connecting binary Boolean function, thus the number of literals in A is less than or equal to $(1-z)n$. The binary Boolean operation joining B and C in the original formula will be referred to as θ.

In the form of the Fundamental Theorem of Boolean Algebra given earlier, for x use $(B\theta C)$; for f_0 use A with a 0 inserted where $(B\theta C)$ had been (with obvious reductions performed); and for f_1 use A with a 1 inserted where $(B\theta C)$ had been (with obvious reductions performed). The "obvious

reduction" of interest is the removal of the constant and the binary Boolean operation to which the constant is input, through appropriate simple transformations. The formula ends up transformed as shown in figure 3.

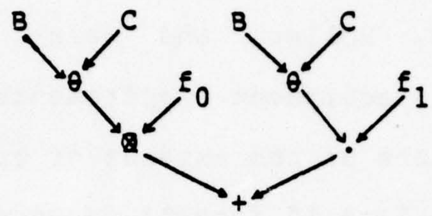

FIGURE 3 TRANSFORMED FORMULA

The value chosen for z results in the maximum possible number of literals in both B and C being upper bounded by .43016 n while the maximum possible number of literals in both f_0 and f_1 is upper bounded by .56984 n. In fact it can be shown that that value for z allows the recursive application of the procedure to result in the previously mentioned value for B2-to-B2 restructuring of 2.465 for R.

1.4 REDUCTION OF EQUIPMENT REQUIREMENTS

The problem here is to devise design methods giving networks with small equipment requirements while still providing "good" depth limits. The techniques used are very similar to those used in section 1.3.

In general, the equipment complexity obtained by these techniques is $O(n^w)$, where n is the number of literals in the original formula and w is a constant determined by the recurrence used.

Preparata, Muller, and Barak [9] obtained methods giving linear equipment requirements for U2-to-AND and -OR gate realizations at the expense of greater network depth. Instead of the "small" formula recurrence used as in section 1.3, "small" networks were used. A network is similar to a formula except that the requirement for a tree structure is relaxed, allowing fan-out in excess of one from the nodes, but not allowing directed cycles.

As an example of how the use of "small" networks can reduce equipment complexity, consider that if the "small" formula in figure 3 (section 1.3.4) is replaced by the "small" network of figure 4, then the necessity for duplication of the B and C subtrees is eliminated, with consequent reduction of equipment requirements.

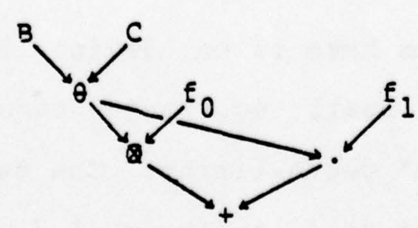

FIGURE 4 "SMALL" NETWORK

1.5 PRELIMINARY INFORMATION

Preparata, Muller, and Barak [9] define the useful concept of composition as follows:

"... an expression E is assumed to be given as a binary tree. Given any subexpression F of E, suppose we replace F in E with a free variable x and let G be the resulting expression. We then define the composition of G and F with respect to x, written $E = G \circ F$, as the expression obtained by substituting F for x in G."

The free variable is not counted in determining the size of G. The free variable can be used to allow an expansion around x for decomposition purposes.

The Fundamental Theorem of Boolean Algebra is used as the decomposition method in this paper and is used both in the Sum of Products form and the Product of Sums form. Also, the complement of these two forms is used in some proofs to obtain the complement of a given expression. If $E = A \circ B$, then the resulting formulas are as follows:

$$E = \bar{B}A_0 + BA_1 = (B + A_0)(\bar{B} + A_1)$$

$$\bar{E} = \bar{B}\bar{A}_0 + B\bar{A}_1 = (B + \bar{A}_0)(\bar{B} + \bar{A}_1)$$

where A_0 is the expression obtained by substituting 0 for B, and A_1 is the expression obtained by substituting 1 for B in E.

The proofs in this paper make frequent use of the fact that any binary Boolean operation, such as $B\theta C$ can be replaced by either $z_1z_2+z_3z_4$ or by $(z_5+z_6)(z_7+z_8)$ with the z_i terms representing B, $\overline{B}$, C, or $\overline{C}$ as required. These replacement networks can be reduced in those cases in which θ is neither EXCLUSIVE-OR nor EQUIVALENCE, so in general the worst case (in terms of depth and equipment required) is encountered in this conversion when θ is one of those two operators.

It is also noted that any expression using operators from B2 can be converted to an equivalent expression, with no change in expression size, using only the operators AND, OR, and EXCLUSIVE-OR. This is done by propagating all complement operations to the literals, through the use of DeMorgan's Laws. Since the literals are assumed to be 2-rail and allow unlimited fan-out, this conversion has no inherent contradiction. To simplify some of the proofs, it will be assumed that this transformation has already been applied to the expression.

All gates in the constructions are assumed to have equal delay with no difference between the delay for 0-1 transitions and the delay for 1-0 transitions. For simplicity, the gate delay time is normalized to 1. The time at which the output from a network is required is defined to be T. For example, in figure 5, for the output

of the network to be available at time T, the input at D must be available no later than time T-1; the input at C, no later than T-2; and the inputs at A and B, no later than T-3.

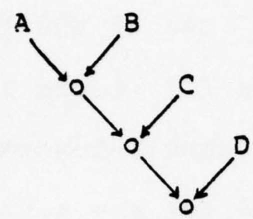

FIGURE 5 TIMING REQUIREMENTS

Note that in all constructions, the complement operation takes precedence over any "prime" type of operation. For example, $\bar{E} = \bar{E}'\bar{E}''$ means that $\bar{E}$ is constructed from the product of two terms, not that E' is produced and then complemented.

1.5.1 LEMMA 2

For restructuring formulas with linear equipment requirements, a lemma by Brent [1] is used (expressed in a form similar to that used in Preparata et al [9]). If G is a Boolean expression with a free variable x and q is any real number such that $1 < q \leq |G|$, then G can be expressed as $A \circ (B\theta C)$ such that $|B\theta C| \geq q$, C contains x, and $|C| < q$. Such a point can be found by following the path from the root of the expression toward x until the subtree

sizes satisfy the conditions.

1.5.2 LEMMA 3

Analysis of equipment requirements is greatly simplified by the use of the following lemma expressed by Muller and Preparata [7] (lemma 5):

> "Let $f(x)$ be a convex-downward function of a real variable x and let $g = a_1 f(x_1)+\ldots+a_r f(x_r)$, where $x_1,\ldots,x_r$ are real nonnegative variables and $a_1,\ldots,a_r$ are positive constants. If the domain of g is the convex hull defined by a set of extreme points, all the maxima of g occur at extreme points."

1.5.3 USING NOR INSTEAD OF NAND

In this paper all procedures for conversions to NAND and NOR networks are expressed in the manner required to produce NAND networks only. Instead of repeating these procedures with the changes necessary for construction of NOR networks, the following procedure is provided for preprocessing and postprocessing networks to allow the procedures for NAND conversions to result in NOR networks:

1. Construct the dual, DUAL(E), of the original expression, E.
2. Apply the appropriate NAND conversion to DUAL(E).

3. Construct the dual of the NAND converted network. (The NAND gates are thus replaced by NOR gates.)

The resulting network is a NOR implementation of the original expression. All bounds found for NAND conversions apply to the corresponding NOR conversions.

CHAPTER 2

FORMULA CONVERSION FOR DEPTH REDUCTION

In this chapter, formula conversion from basis B2 to NAND will be described for gate fan-in limits of 2, 3, and 4. For the 2-input case, the depth upper bound of $2.88 \log |E| + O(1)$, which was obtained by McColl [6] will be described. The same bound will then be met using another construction method which is an extension of work by Pratt [8]. The conjecture that the above bound cannot be improved will be presented. A construction will then be demonstrated which, for the 3-input case, gives a depth upper bound of $2 \log |E| + O(1)$. Finally, a construction for the 4-input case with a depth upper bound of $1.44 \log |E| + O(1)$ will be given.

With this particular construction, the results will be represented in terms of AND and OR gates which will alternate from the OR gate at the output, thus leaving a simple conversion to NAND gates for the end. In some instances in these constructions, an AND gate with a constant one input term and an OR gate with a constant zero input term is used to force the required alternation.

2.1 MCCOLL'S RESULT [6] ON B2 TO 2-INPUT NAND

McColl's bound is presented here, but in a style more like that used by Preparata et al [9],[10].

Any Boolean formula over the basis B2 can be restructured so that it describes a network of NAND gates with maximum fan-in of 2 and with depth bounded from above by

$$(-2/\log \beta) \log n + O(1) \approx 2.88 \log n + O(1)$$

where n = the number of literals in the formula and β is the positive root of $z^2 + z - 1 = 0$. ($\beta \approx .6180$.)

Proof:

Let E be a Boolean expression with $|E| \geq 1$.

Define t(E) to be the minimum depth of any network realizing the function represented by E.

Assume inductively:

P1: $t(E) \leq (-2/\log \beta) \log |E| + K$

Clearly, by the choice of a large enough value for K, the Basis step can be satisfied for the induction starting with n=3.

The extension of P1 is provided by the following algorithm, where $|E| = n$.

STEP 1: Using Lemma 1, decompose E as

$A \circ (B\theta C)$ with

$|B| \leq |C| < \beta^2 n$

$|A| = |E| - |B| - |C| \leq n - \beta^2 n = \beta n$

since $\beta^2 + \beta - 1 = 0$ giving $\beta = 1 - \beta^2$

STEP 2: By the Fundamental Theorem of Boolean Algebra

$$E = \overline{(B\theta C)}A_0 + (B\theta C)A_1$$

But $\overline{B\theta C}$ can be expressed as $D_1D_2+D_3D_4$ and $B\theta C$ can be expressed similarly, where each D_i term corresponds to B, $\overline{B}$, C, or $\overline{C}$.

$$E = (D_1D_2+D_3D_4)A_0+(D_5D_6+D_7D_8)A_1$$

By P1 the D_i type terms are of depth

$$t(D_i) \leq (-2/\log \beta) \log(\beta^2 n) + K$$
$$= (-2/\log \beta) \log n + K - 4$$

Thus, the depth bound for D_i type terms is 4 less than the hypothesized depth bound for E. But the latest time at which D_i type terms in the expression must be available is T-4, (recall that T is defined to be the time at which the output from the construction is required) so there is no depth contradiction.

By P1 the A_0 type terms are of depth

$$t(A_0) \leq (-2/\log \beta)\ \log(\ \beta n) + K$$
$$= (-2/\log \beta)\ \log n + K - 2$$

Thus, the depth bound for A_0 type terms is 2 less than the hypothesized depth bound for E. But the latest time at which A_0 type terms in the expression must be available is T-2, so there is no depth contradiction.

All variables used in the P1 decomposition fall into one of these two categories, so there is no contradiction in the P1 decomposition. Note that in no case has a fan-in greater than 2 been used and that the computation tree produced has an OR gate at the root with every path from the root consisting of alternating AND and OR gates, thus allowing a representation consisting only of NAND gates.

2.2 EXTENSION OF PRATT'S WORK TO B2 TO 2-INPUT NAND CONVERSION

In "The Effect of Basis on Size of Boolean Expressions" by V.R. Pratt [8], a restructuring technique and analysis results are presented. This work is summarized below.

2.2.1 PRATT'S RESULTS

1. The expression is split into two expressions, one of which is a subexpression, X, of the original expression, with the property that that subexpression

has as close as possible to 1/2 the number of literals in the original expression. Then the Fundamental Theorem of Boolean Algebra is applied giving

$$E = \overline{X}A_0 + XA_1$$

where A_0 and A_1 are obtained by substituting 0 and 1 respectively for X in the original expression.

The technique is then recursively applied to X, $\overline{X}$, A_0, and A_1.

2. Let n be the number of literals in the original expression.

Let pn and qn be the number of literals in the two pieces into which the original expression was separated, with $q \leq p$. The resulting rearrangement has size bounds as shown in figure 6.

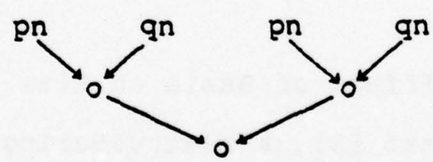

FIGURE 6 FIRST APPLICATION

3. Now, consider a repeated application of the procedure in 1. to the larger (pn sized) segment. This results in two more segment size bounds, rn and sn, with $s \leq r$. The resulting rearrangement has size bounds as shown in figure 7.

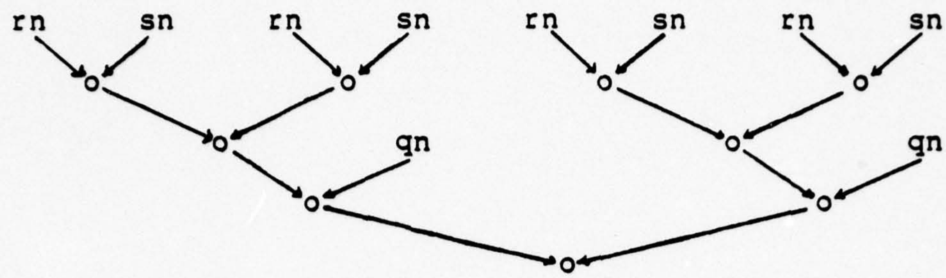

FIGURE 7 SECOND APPLICATION

4. Pratt found the following inequalities to constrain r, q, and s:

(i) $r+s+q = 1$

(ii) $q \leq r+s$ $\quad 1/2 \leq r+s$

(iii) $s \leq r$

(iv) $r \leq 2s$

(v) $r \leq q$

5. Graphically, r and s are constrained to be within the quadrilateral in figure 8.

2.2.2 EXPLANATION OF PRATT'S CONSTRAINTS

The first 3 of Pratt's constraints are obvious, however, the last two require some explanation.

Constraint (iv), $r \leq 2s$, can be shown to be true by using Lemma 1 to decompose E as:

$A \circ (B\theta C)$ with

$|B| \leq |C| < 1/3\ n$ giving $|B\theta C| < 2/3\ n$

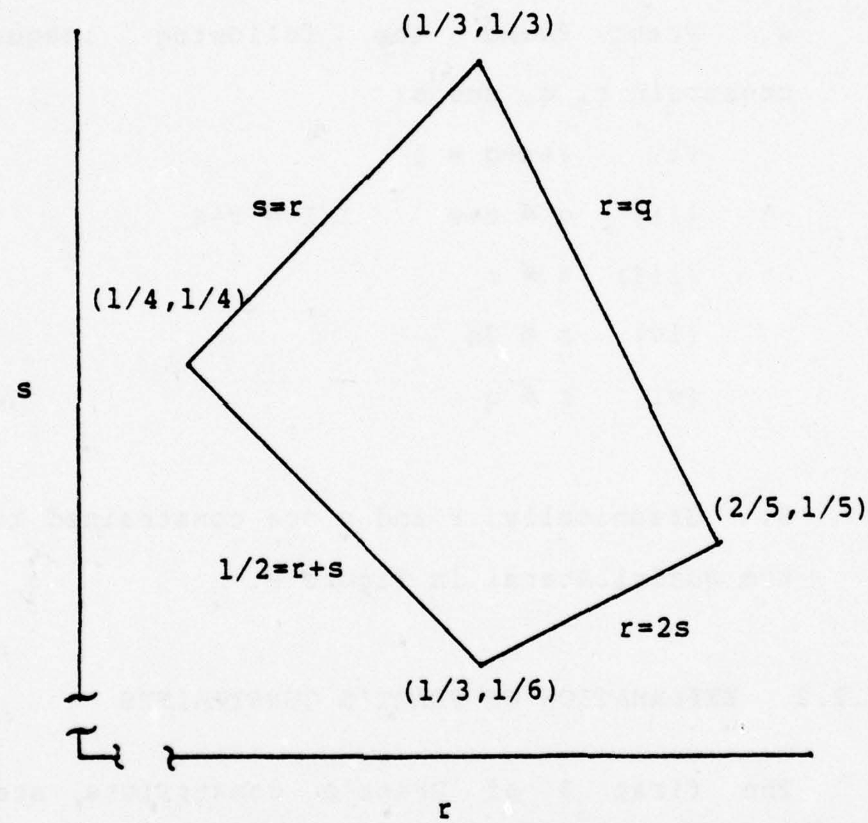

FIGURE 8 PRATT'S CONSTRAINT GRAPH

$$|A| = |E| - |B\theta C| \leq 2/3\ n$$

This holds for $n > 3$, but examination of the case $n = 3$ also shows that it is always possible to separate an expression with $n > 2$ into two components, A and $(B\theta C)$, such that each of them is of size less than or equal to $2/3\ n$. The worst case would be where one of the components has size $1/3\ n$ and the other, size $2/3\ n$. Since s is defined to be less than r, the bound $r \leq 2s$ is shown to hold.

Constraint (v), $r \leq q$, is somewhat more complex to show.

For the decomposition $E = A \circ (B\theta C)$, if $|A| \leq |B\theta C|$ (see figure 9), then note that $|B| \leq |A|$ and $|C| \leq |A|$, since otherwise a better (closer to $1/2\ n$) decomposition could have been obtained by choosing to make the separation between θ and the larger of B and C. The next decomposition can do no worse than to separate B and C giving $r \leq q$.

If $|A| > |B\theta C|$ (see figure 10), then define $H = (B\theta C)$ and $E = G \circ (H\phi M)$. Now $|G| \leq |H|$ and $|M| \leq |H|$, since otherwise a better decomposition could have been obtained by choosing to make the separation split off the larger of G and M. The next decomposition can do no worse than to separate G and M, giving $r \leq q$.

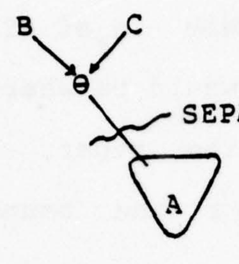

FIGURE 9 CONSTRAINT V SEPARATION ("A" SMALL)

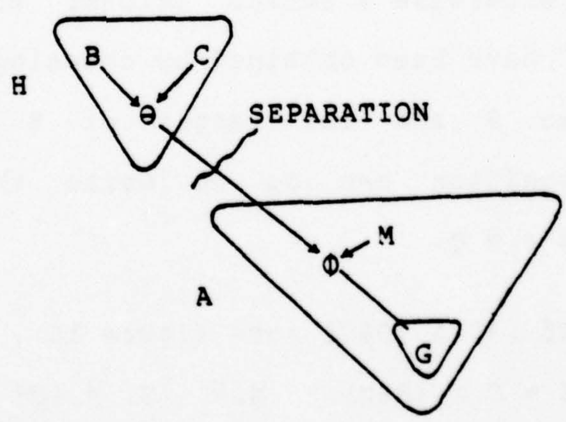

FIGURE 10 CONSTRAINT V SEPARATION ("A" LARGE)

2.2.3 EXTENSION OF PRATTS WORK TO 2-INPUT NAND DEPTH

Now, using the preceding results by Pratt, the depth of a network constructed by that method will be shown to be bounded from above by

$$2.88 \log n + O(1)$$

First it must be noted that for construction of X, for use in the Fundamental Theorem of Boolean Algebra, with complemented form of the input literals available, the depth is the same as the depth of X since the complement operation can be propagated to the literals by using DeMorgan's Laws.

Let β be the positive root of $z^2 + z - 1 = 0$. ($\beta \approx .6180$.)

Now consider Pratt's graph again, with the $r+s = \beta$ line included, as shown in figure 11.

Two cases will be considered;

CASE 1: $r+s < \beta$

CASE 2: $r+s \geq \beta$

Define $V = -2/\log \beta \approx 2.88$.

Let $t(|E|)$ be the minimum depth of the deepest network required to realize the function represented by any expression E with $|E|$ literals.

Now assume inductively that for $|E| < n$

$$t(|E|) \leq V \log |E| + W$$

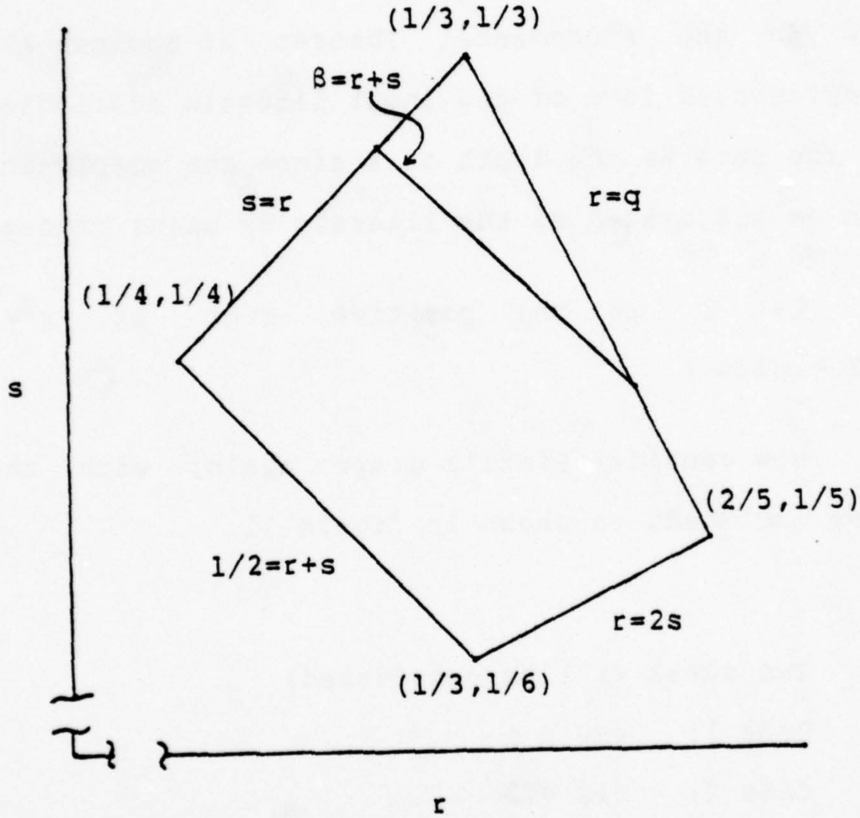

FIGURE 11 MODIFIED CONSTRAINT GRAPH

Clearly, W can be chosen large enough to satisfy the basis step.

CASE 1: $r+s < \beta$

$p < \beta$ since $p = r+s$

Note that p and q must be available at time T-2 in the construction to allow output at time T.

Recall that $q \leq p$, thus $t(qn) \leq t(pn)$

$t(pn) \leq V \log pn + W$

$t(pn) < V \log \beta n + W$

$t(pn) < V \log n + W + V \log \beta$

but $V = -2/\log \beta$

$t(pn) < V \log n + W - 2$

Thus the depth bound for pn bounded terms is 2 less than the hypothesized depth bound for E. But the latest time at which the pn bounded terms in the expression must be available is T-2, so there is no depth contradiction.

Also note that since $q \leq p$, there is no contradiction with the qn bounded terms.

Thus, there is no contradiction in case 1.

CASE 2: $r+s \geq \beta$

But $r+s+q = 1$

thus $q \leq 1-\beta$.

Note that r and s must be available at time T-4 in the construction.

Recall that $s \leq r \leq q$

thus $t(sn) \leq t(rn) \leq t(qn)$.

$$t(qn) \leq V \log qn + W$$

$$t(qn) \leq V \log ((1-\beta)n) + W$$

$$t(qn) \leq V \log n + W + V \log (1-\beta)$$

$$V = -2/\log \beta = -4/\log \beta^2$$

but $\beta^2 + \beta - 1 = 0$ thus $\beta^2 = 1 - \beta$

thus $V = -4/\log(1-\beta)$

$$t(qn) \leq V \log n + W - 4$$

Thus the depth bound for qn bounded terms is 4 less than the hypothesized depth bound for E. But the latest time at which the qn bounded terms in the expression must be available is T-2, so there is no depth contradiction.

Also note that since r and s are less than or equal to q, the depth bound for rn and sn bounded terms is also 4 less than the hypothesized depth bound for E. But the latest time at which the rn and sn bounded terms in the expression must be available is T-4, so there is no depth contradiction.

Thus, there is no contradiction in case 2.

Note that in no case has a fan-in greater than 2 been used and that the computation tree produced has an OR gate at the root with every path from the root consisting of alternating AND and OR gates, thus allowing a representation consisting only of NAND gates.

2.3 ON THE CONJECTURE THAT B2 TO 2-INPUT NAND CONVERSION CANNOT BE DONE WITH A TIGHTER DEPTH UPPER BOUND THAN 2.88 LOG |E| + O(1)

The conjecture is proposed that B2 to 2-input NAND conversion cannot be done with a better depth upper bound than 2.88 log |E| + O(1).

As support for this conjecture, a construction method is given for a sequence of expressions, E_i, which have computation structures such that they do not appear to be restructurable to reduce their depth below 2.88 log |E| - C for some constant C.

The construction is as follows:

1) $E_0 = a$ (a single variable)

2) $E_1 = (b \oplus c)d$

3) $E_2 = ((e \oplus f) \oplus g)h$

4) $E_j = ((E_{j-2} \oplus E_{j-3}) \oplus E'_{j-2})a_j$

It is understood that all variables appearing in expression E_j are distinct. (eg. The only difference between E_{j-2} and E'_{j-2} is that they are constructed from distinct variables.)

E_j is shown in figure 12.

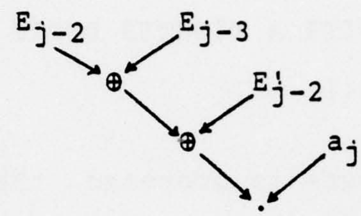

FIGURE 12 E_j STRUCTURE

2.4 B2 TO 3-INPUT NAND CONVERSION

Any Boolean formula over the basis B2 can be restructured so that it corresponds to a network of NAND gates with maximum fan-in of 3 with depth bounded from above by

$$2 \log n + O(1)$$

where n = the number of literals in the formula.

Proof:

Let E be a Boolean expression with $|E| \geq 1$.

Define t(E) to be the minimum depth of any network realizing the function represented by E.

Define $t_2(E)$ to be the minimum depth of any network with outputs E' and E" such that E' E" is equivalent to E.

Assume inductively:

P1: $t(E) \leq 2 \log |E| + K$

P2: $t_2(E) \leq 2 \log |E| + 2 \log 3 - 4 + K$

Clearly, by the choice of a large enough value for K, the Basis step can be satisfied for the induction starting with n=4.

The extension of P1 is provided by the following algorithm, where $|E| = n$.

STEP 1: Using Lemma 1, decompose E as

$A \circ (B\theta C)$ with

$|B| \leq |C| < 1/3\ n$

$|A| = |E| - |B| - |C| \leq n - 1/3\ n = 2/3\ n$

By the Fundamental Theorem of Boolean Algebra

$E = \overline{(B\theta C)}A_0 + (B\theta C)A_1$

STEP 2: If $|B| \leq 1/4\ n$

then for θ = EXCLUSIVE-OR, express E as

$E = (BC+\bar{B}\bar{C})A_0 + (B\bar{C}+\bar{B}C)A_1$

$E = (BC'C''+\bar{B}\bar{C}'\bar{C}'')A_0'A_0'' + (B\bar{C}'\bar{C}''+\bar{B}C'C'')A_1'A_1''$

where $C = C'C''$ and $\bar{C} = \bar{C}'\bar{C}''$. (Recall that primes are

performed <u>after</u> the complement operation.)

Note that for θ = AND and OR, the terms $(B\theta C)$ could be expressed in the form $(BC'C''+0)$ and $(B1+C'C'')$, which is no worse than the θ = EXCLUSIVE-OR case, so the analysis will be done for EXCLUSIVE-OR.

By P1 the B type terms are of depth

$$t(B) \leq 2 \log(\ 1/4\ n) + K$$
$$= 2 \log n + K - 4$$

Thus, the depth bound for B type terms is 4 less than the hypothesized depth bound for E. But the latest time at which B type terms in the expression must be available is T-4, (recall that time T is the time at which the output is required) so there is no depth contradiction.

By P2 the C' type terms are of depth

$$t_2(C) \leq 2 \log(\ 1/3\ n) + 2 \log 3 - 4 + K$$
$$= 2 \log n + K - 4$$

Thus, the depth bound for C' type terms is 4 less than the hypothesized depth bound for E. But the latest time at which C' type terms in the expression must be available is T-4, so there is no depth contradiction.

By P2 the A_0' type terms are of depth

$$t_2(A_0) \leq 2 \log(\ 2/3\ n) + 2 \log 3 - 4 + K$$
$$= 2 \log n + K - 2$$

Thus, the depth bound for A_0' type terms is 2 less than the hypothesized depth bound for E. But the latest time at which A_0' type terms in the expression must be available is T-2, so there is no depth contradiction.

All variables used in the step 2 decomposition fall into one of these three categories, so there is no contradiction in the step 2 decomposition.

STEP 3: If $|B| > 1/4\ n$

then $|A| = |E| - |B| - |C| < n - 1/4\ n - 1/4\ n = 1/2\ n$

For θ = EXCLUSIVE-OR, express E as

$$E = (B+\overline{C})(\overline{B}+C)A_0+(B+C)(\overline{B}+\overline{C})A_1$$

$$E = (B'B''+\overline{C}'\overline{C}'')(\overline{B}'\overline{B}''+C'C'')A_0$$
$$+(B'B''+C'C'')(\overline{B}'\overline{B}''+\overline{C}'\overline{C}'')A_1$$

Note that for θ = AND and OR, the terms $(B\theta C)$ could be expanded in the form $(B'B''+0)(C'C''+0)$ and $(B'B''+C'C'')$, which is no worse than the θ = EXCLUSIVE-OR case, so the analysis will be done for EXCLUSIVE-OR.

The B' type and the C' type terms are both similar to the C' type terms in step 2 and the same arguments apply, so there is no contradiction with them.

By P1 the A_0 type terms are of depth

$$t(A_0) \leq 2\log(\ 1/2\ n) + K$$
$$= 2\log n + K - 2$$

Thus, the depth bound for A_0 type terms is 2 less than the hypothesized depth bound for E. But the latest time at which A_0 type terms in the expression must be available is T-2, so there is no depth contradiction.

All variables used in the step 3 decomposition fall into one of these three categories, so there is no contradiction in the step 3 decomposition.

Thus there is no contradiction in the P1 decomposition.

The extension of P2 is provided by the following algorithm, where $|E| = n$. (E is restructured as E'E".)

STEP 1: Using Lemma 1, decompose E as

$A \circ (B \theta C)$ with

$|B| \leq |C| < 1/4\ n$

$|A| = |E| - |B| - |C| \leq n - 1/4\ n = 3/4\ n$

By the Fundamental Theorem of Boolean Algebra

$E = ((B\theta C)+A_0)(\overline{(B\theta C)}+A_1)$

Now let $E' = (B\theta C)+A_0$ and $E" = \overline{(B\theta C)}+A_1$.

Since E' and E" are similar, further arguments will be directed to E' as representative of both E' and E".

But $B\theta C$ can be expressed as $(D_1+D_2)(D_3+D_4)$, where each D_i term corresponds to B, $\overline{B}$, C, or $\overline{C}$.

$E' = (D_1+D_2)(D_3+D_4)+A_0$

$$E' = (D_1'D_1''+D_2'D_2'')(D_3'D_3''+D_4'D_4'')+A_0$$

By P2 the D_i' type terms are of depth

$$t_2(D_i) \leq 2\log(1/4\ n) + 2\log 3 - 4 + K$$
$$= 2\log n + 2\log 3 - 4 + K - 4$$

Thus, the depth bound for D_i' type terms is 4 less than the hypothesized depth bound for E'. But the latest time at which D_i' type terms in the expression must be available is T-4, so there is no depth contradiction.

Now define

$$H = (D_1'D_1''+D_2'D_2'')(D_3'D_3''+D_4'D_4'')$$
$$E' = H+A_0$$

STEP 2: Using Lemma 1, decompose A_0 as $G \circ (Q \phi R)$ with

$$|Q| \leq |R| < 1/3\ |A_0| \leq 1/4\ n$$
$$|G| \leq 2/3\ |A_0| \leq 1/2\ n$$

STEP 3: If $|Q| \leq 3/16\ n$ then for ϕ = EXCLUSIVE-OR, express E as

$$E' = H+(QR+\bar{Q}\bar{R})G_0+(Q\bar{R}+\bar{Q}R)G_1$$
$$E' = H+(QR'R''+\bar{Q}\bar{R}'\bar{R}'')G_0'G_0''+(Q\bar{R}'\bar{R}''+\bar{Q}R'R'')G_1'G_1''$$

Note that for ϕ = AND and OR, the terms $(Q\phi R)$ could be expressed in the form (QR'R"+0) and (Q1+R'R"), which is no worse than the ϕ = EXCLUSIVE-OR case, so the analysis will be done for EXCLUSIVE-OR.

By P1 the Q type terms are of depth

$$t(Q) \leq 2\log(3/16\ n) + K$$
$$= 2\log n + 2\log 3 - 4 + K - 4$$

Thus, the depth bound for Q type terms is 4 less than the hypothesized depth bound for E'. But the latest time at which Q type terms in the expression must be available is T-4, so there is no depth contradiction.

By P2 the R' type terms are of depth

$$t_2(R) \leq 2\log(1/4\ n) + 2\log 3 - 4 + K$$
$$= 2\log n + 2\log 3 - 4 + K - 4$$

Thus, the depth bound for R' type terms is 4 less than the hypothesized depth bound for E'. But the latest time at which R' type terms in the expression must be available is T-4, so there is no depth contradiction.

By P2 the G_0' type terms are of depth

$$t_2(G_0) \leq 2\log(1/2\ n) + 2\log 3 - 4 + K$$
$$= 2\log n + 2\log 3 - 4 + K - 2$$

Thus, the depth bound for G_0' type terms is 2 less than the hypothesized depth bound for E'. But the latest time at which G_0' type terms in the expression must be available is T-2, so there is no depth contradiction.

All variables used in the step 3 decomposition fall into one of these three categories, so there is no contradiction in the step 3 decomposition.

STEP 4: If $|Q| > 3/16\ n$

then $|G| = |A_0| - |Q| - |R| < 3/4\ n - 3/16\ n - 3/16\ n$

giving $|G| < 3/8\ n$

For ϕ = EXCLUSIVE-OR, express E as

$$E' = H+(Q+\bar{R})(\bar{Q}+R)G_0+(Q+R)(\bar{Q}+\bar{R})G_1$$

$$E' = H+(Q'Q''+\bar{R}'\bar{R}'')(\bar{Q}'\bar{Q}''+R'R'')G_0$$
$$+(Q'Q''+R'R'')(\bar{Q}'\bar{Q}''+\bar{R}'\bar{R}'')G_1$$

Note that for ϕ = AND and OR, the terms $(Q\phi R)$ could be expanded in the form $(Q'Q''+0)(R'R''+0)$ and $(Q'Q''+R'R'')$, which is no worse than the ϕ = EXCLUSIVE-OR case, so the analysis will be done for EXCLUSIVE-OR.

The Q' type and the R' type terms are both similar to the R' type terms in step 3 and the same arguments apply, so there is no contradiction with them.

By P1 the G_0 type terms are of depth

$$t(G_0) \leq 2\log(\ 3/8\ n) + K$$
$$= 2\log n + 2\log 3 - 4 + K - 2$$

Thus, the depth bound for G_0 type terms is 2 less than the hypothesized depth bound for E'. But the latest time at which G_0 type terms in the expression must be

available is T-2, so there is no depth contradiction.

All variables used in the step 4 decomposition fall into one of these three categories, so there is no contradiction in the step 4 decomposition.

Thus there is no contradiction in the P2 decomposition.

Note that in no case has a fan-in greater than 3 been used and that the computation tree produced has an OR gate at the root with every path from the root consisting of alternating AND and OR gates, thus allowing a representation consisting only of NAND gates.

The recurrences used in this construction can be represented as a directed graph (henceforth called the implication graph) with nodes indicating the recurrence formulas (P1, P2, etc.) and an edge from node X to node Y indicating that in node X's recurrence, the node Y recurrence is used. For this construction, the implication graph is the complete graph on two nodes.

2.5 B2 TO 4-INPUT NAND CONVERSION

Any Boolean formula over the basis B2 can be restructured so that it describes a network of NAND gates with maximum fan-in of 4 and with depth bounded from above by

$$(-1/\log \beta) \log n + O(1) \approx 1.44 \log n + O(1)$$

where n = the number of literals in the formula and β is the positive root of $z^2 + z - 1 = 0$. ($\beta \approx .6180$.)

Proof:

Let E be a Boolean expression with $|E| \geq 1$.

Define t(E) to be the minimum depth of any network realizing the function represented by E.

Define $t_2(E)$ to be the minimum depth of any network with outputs E' and E" such that E' E" is equivalent to E.

Define $t_3(E)$ to be the minimum depth of any network with outputs E"' and E"" such that E"'+E"" is equivalent to E.

Assume inductively:

P1: $t(E) \leq (-1/\log \beta) \log |E| + K$

P2: $t_2(E) \leq (-1/\log \beta) \log |E| - 1 + K$

P3: $t_3(E) \leq (-1/\log \beta) \log |E| - 1 + K$

Clearly, by the choice of a large enough value for K, the Basis step can be satisfied for the induction starting with n=3.

The extension of P1 is provided by the following algorithm, where $|E| = n$.

STEP 1: Using Lemma 1, decompose E as

$A \circ (B\theta C)$ with

$|B| \leq |C| < \beta^2 n$

$|A| = |E| - |B| - |C| \leq n - \beta^2 n = \beta n$

since $\beta^2 + \beta - 1 = 0$ giving $\beta = 1 - \beta^2$

STEP 2: By the Fundamental Theorem of Boolean Algebra

$E = \overline{(B\theta C)}A_0 + (B\theta C)A_1$

But $\overline{B\theta C}$ can be expressed as $D_1D_2+D_3D_4$ and $B\theta C$ can be expressed similarly, where each D_i term corresponds to B, $\overline{B}$, C, or $\overline{C}$.

$$E = (D_1D_2+D_3D_4)A_0+(D_5D_6+D_7D_8)A_1$$

$$E = D_1D_2A_0'A_0''+D_3D_4A_0'A_0''+D_5D_6A_1'A_1''+D_7D_8A_1'A_1''$$

By P1 the D_i type terms are of depth

$$t(D_i) \leq (-1/\log \beta)\log(\beta^2 n) + K$$
$$= (-1/\log \beta)\log n + K - 2$$

Thus, the depth bound for D_i type terms is 2 less than the hypothesized depth bound for E. But the latest time at which D_i type terms in the expression must be available is T-2, so there is no depth contradiction.

By P2 the A_0' type terms are of depth

$$t_2(A_0) \leq (-1/\log \beta)\log(\beta n) - 1 + K$$

$$=(-1/\log \beta) \log n + K - 2$$

Thus, the depth bound for A_0' type terms is 2 less than the hypothesized depth bound for E. But the latest time at which A_0' type terms in the expression must be available is T-2, so there is no depth contradiction.

All variables used in the P1 decomposition fall into one of these two categories, so there is no contradiction in the P1 decomposition.

The extension of P2 is provided by the following algorithm, where $|E| = n$. (E is restructured as E'E".)

STEP 1: Using Lemma 1, decompose E as

$A \circ (B \theta C)$ with

$|B| \leq |C| < \beta^2 n$

$|A| = |E| - |B| - |C| \leq n - \beta^2 n = \beta n$

since $\beta^2 + \beta - 1 = 0$ giving $\beta = 1 - \beta^2$

STEP 2: By the Fundamental Theorem of Boolean Algebra

$E = ((B\theta C)+A_0)(\overline{(B\theta C)}+A_1)$

Now let $E' = (B\theta C)+A_0$ and $E'' = \overline{(B\theta C)}+A_1$.

Since E' and E" are similar, further arguments will be directed to E' as representative of both E' and E".

But $B\theta C$ can be expressed as $D_1D_2+D_3D_4$, where each D_i term corresponds to B, $\bar{B}$, C, or $\bar{C}$.

$$E' = D_1D_2+D_3D_4+A_0$$

STEP 3: Now, again use lemma 1, with identical parameters, to decompose A_0. Again use the Fundamental Theorem of Boolean Algebra (expressed in its sum of products form) to express A_0 as $(G_1+G_2)(G_3+G_4)H_0+(G_5+G_6)(G_7+G_8)H_1$, where the G_i terms correspond to the new B, $\bar{B}$, C, or $\bar{C}$.

$$|G| \leq \beta^2|A_0| \leq \beta^3 n$$

$$|H| \leq \beta\ |A_0| \leq \beta^2 n$$

$$E' = D_1D_2+D_3D_4+(G_1+G_2)(G_3+G_4)H_0$$
$$+(G_5+G_6)(G_7+G_8)H_1$$

$$E' = D_1'D_1''D_2'D_2''+D_3'D_3''D_4'D_4''$$
$$+(G_1''{}'+G_1''{}''+G_2''{}'+G_2''{}'')$$
$$(G_3''{}'+G_3''{}''+G_4''{}'+G_4''{}'')H_0'H_0''$$
$$+(G_5''{}'+G_5''{}''+G_6''{}'+G_6''{}'')$$
$$(G_7''{}'+G_7''{}''+G_8''{}'+G_8''{}'')H_1'H_1''$$

By P2 the D_i' type terms are of depth

$$t_2(D_i) \leq (-1/\log\beta)\log(\beta^2 n) - 1 + K$$
$$=(-1/\log\beta)\log n - 1 + K - 2$$

Thus, the depth bound for D_i' type terms is 2 less than the hypothesized depth bound for E'. But the latest time at which D_i' type terms in the expression must be available is T-2, so there is no depth contradiction.

By P3 the $G_i''{}'$ type terms are of depth

$$t_3(G_i) \leq (-1/\log\beta)\log(\beta^3 n) - 1 + K$$

$$=(-1/\log \beta) \log n - 1 + K - 3$$

Thus, the depth bound for G_i''' type terms is 3 less than the hypothesized depth bound for E'. But the latest time at which G_i''' type terms in the expression must be available is T-3, so there is no depth contradiction.

By P2 the H_0' type terms are of depth

$$t_2(H_0) \leq (-1/\log \beta) \log(\beta^2 n) - 1 + K$$
$$=(-1/\log \beta) \log n - 1 + K - 2$$

Thus, the depth bound for H_0' type terms is 2 less than the hypothesized depth bound for E'. But the latest time at which H_0' type terms in the expression must be available is T-2, so there is no depth contradiction.

All variables used in the P2 decomposition fall into one of these three categories, so there is no contradiction in the P2 decomposition.

The extension of P3 is provided by the following algorithm, where $|E| = n$. (E is restructured as E"'+E"".)

STEP 1: Using Lemma 1, decompose E as

$A \circ (B \theta C)$ with

$|B| \leq |C| < \beta^2 n$

$|A| = |E| - |B| - |C| \leq n - \beta^2 n = \beta n$

since $\beta^2 + \beta - 1 = 0$ giving $\beta = 1 - \beta^2$

STEP 2: By the Fundamental Theorem of Boolean Algebra

$$E = \overline{(B\theta C)}A_0 + (B\theta C)A_1$$

Now let $E''' = \overline{(B\theta C)}A_0$ and $E'''' = (B\theta C)A_1$.

Since E"' and E"" are similar, further arguments will be directed to E"' as representative of both E"' and E"".

But $\overline{B\theta C}$ can be expressed as $(D_1+D_2)(D_3+D_4)$ where each D_i term corresponds to B, $\bar{B}$, C, or $\bar{C}$.

$$E''' = (D_1+D_2)(D_3+D_4)A_0$$

$$E''' = (D_1''' + D_1'''' + D_2''' + D_2'''')$$
$$(D_3''' + D_3'''' + D_4''' + D_4'''')A_0'A_0''$$

By P3 the D_i''' type terms are of depth

$$t_3(D_i) \leq (-1/\log \beta) \log(\beta^2 n) - 1 + K$$
$$= (-1/\log \beta) \log n - 1 + K - 2$$

Thus, the depth bound for D_i''' type terms is 2 less than the hypothesized depth bound for E"'. But the latest time at which D_i''' type terms in the expression must be available is T-2, so there is no depth contradiction.

By P2 the A_0' type terms are of depth

$$t_2(A_0) \leq (-1/\log \beta) \log(\beta n) - 1 + K$$
$$= (-1/\log \beta) \log n - 1 + K - 1$$

Thus, the depth bound for A_0' type terms is 1 less than the hypothesized depth bound for E"'. But the latest time at which A_0' type terms in the expression must be available is T-1, so there is no depth contradiction.

All variables used in the P3 decomposition fall into one of these two categories, so there is no contradiction in the P3 decomposition.

Note that in no case has a fan-in greater than 4 been used and that the computation tree produced has an OR gate at the root with every path from the root consisting of alternating AND and OR gates, thus allowing a representation consisting only of NAND gates.

The implication graph for this construction is shown in figure 13.

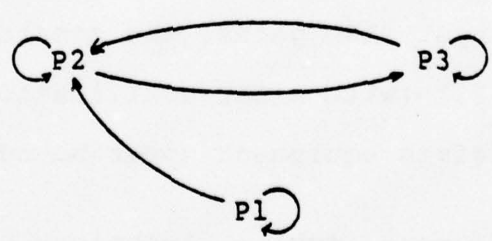

FIGURE 13 IMPLICATION GRAPH

CHAPTER 3

EQUIPMENT COMPLEXITY CORRESPONDING TO RESTRUCTURED FORMULAE

In this chapter, the amount of equipment (in number of gates) as a function of the input formula size is examined for constructions intended to give good depth reduction. (Depth is not sacrificed to improve equipment requirements.) Interest is focused on the order of equipment growth.

The techniques used are modeled after those used by Preparata et al.[9],[10] and also a technique used by Muller and Preparata [7].

In general, the growth of equipment for these constructions is upper bounded by $O(|E|^{w})$, where w is determined by the construction used.

For 2-input NAND gates, the construction method given in section 2.1 (with minor modification to reduce equipment complexity) gives equipment upper bounded by $O(|E|^{1.832})$.

A much more subtle technique, which is a minor modification of the technique used by Muller and Preparata [7], results in equipment upper bounded by $O(|E|^{1.44})$ for 2-input NAND gates.

For 3-input NAND gates, the construction method given in section 2.4 (with minor modification to reduce equipment complexity) gives equipment upper bounded by $O(|E|^{2.216})$.

For 4-input NAND gates, the construction method given in section 2.5 (with minor modification to reduce equipment complexity) gives equipment upper bounded by $O(|E|^{1.832})$.

3.1 2-INPUT NAND EQUIPMENT COMPLEXITY, METHOD 1

The technique to obtain an upper bound on 2-input NAND equipment complexity of $O(|E|^{1.832})$ is presented here.

To reduce equipment complexity, the basic construction of McColl [6], described in section 2.1, is modified to produce both E and $\overline{E}$ by using the equations noted in section 1.5:

$$E = \overline{(B\theta C)}A_0 + (B\theta C)A_1$$

$$\overline{E} = \overline{(B\theta C)}\overline{A}_0 + (B\theta C)\overline{A}_1$$

thus $\overline{E}$ can be produced merely by replacing A_0 and A_1 by $\overline{A}_0$ and $\overline{A}_1$, respectively, in the network producing E. The construction is then given as in figure 14, where the z_i terms represent B, $\overline{B}$, C, or $\overline{C}$ as required.

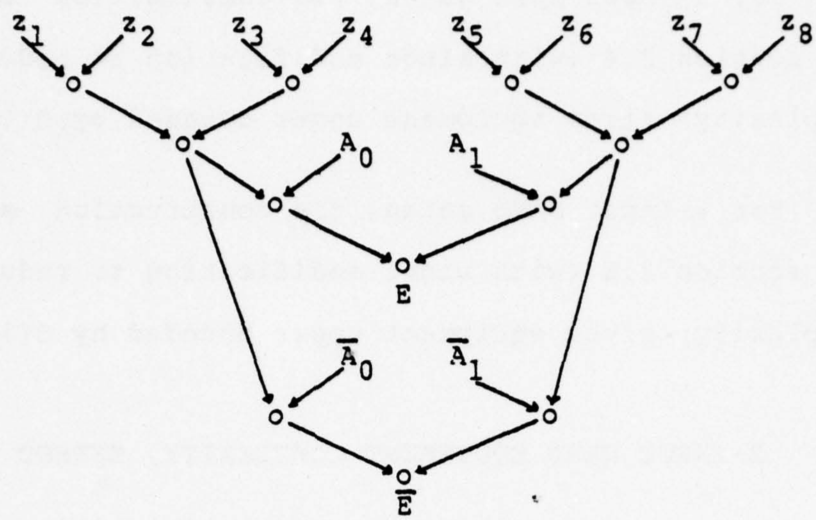

FIGURE 14 2-INPUT GATE NETWORK

ASSERTION:

Using the construction given in section 2.1, modified as described above, the equipment required is upper bounded by $O(|E|^{1.832})$.

PROOF:

Let the number of gates used in the construction of E and $\overline{E}$ together be expressed as Y(E).

Inductively assume that for $|E| < n$

$$Y(E) < K(|E|^{w}-1)$$

where $w = (\log(\sqrt{2}-1))/\log \beta \simeq 1.832$,

and β is the positive root of $z^2+z-1 = 0$.

($\beta \simeq .6180$.)

Clearly, by the choice of large enough values for K, the Basis step can be satisfied for the induction.

Examination of the recursive application of the construction in figure 14 gives

$$Y(E) = Y(A_0)+Y(A_1)+Y(B)+Y(C)+12$$
$$< K(|A_0|^w-1)+K(|A_1|^w-1)+K(|B|^w-1)+K(|C|^w-1)+12$$

Note that Lemma 3 (section 1.5.2) applies since $K(y^w-1)$ is a convex downward function of y, K is a positive constant, and y is a real nonnegative variable in this context.

By examining the extreme points of the domain of $|A|$, $|B|$, and $|C|$ it is easy to see that the maximum number of gates required is obtained when

$$|A| = n\beta\ , \quad |B| = 1, \quad \text{and} \quad |C| = n\,\beta^2-1$$

thus $Y(E) < 2K((n\beta\ )^w-1)+K((n\ \beta^2-1)^w-1)+12$

$$< K(n^w(2\ \beta^w+\ \beta^{2w})-1)-2K+12$$

Now note that if K is chosen to be larger than or equal to 6, then -2K+12 is less than or equal to zero.

thus $Y(E) < K(n^w(2\ \beta^w+\ \beta^{2w})-1)$

when $K \geq 6$

Now note that $2X+X^2 = 1$ when $X = \sqrt{2}-1$

Thus by substituting β^w for X,

$$2\ \beta^w+\ \beta^{2w} = 1 \quad \text{when} \quad \beta^w = \sqrt{2}-1$$

but $\beta^w = \sqrt{2}-1$ when $w \log \beta = \log(\sqrt{2}-1)$

or when $w = (\log(\sqrt{2}-1))/\log \beta \approx 1.832$

thus $Y(E) < K(n^w-1)$ when $w = (\log(\sqrt{2}-1))/\log \beta$ and $K \geq 6$.

3.2 2-INPUT NAND EQUIPMENT COMPLEXITY, METHOD 2

The technique to obtain an upper bound on 2-input NAND equipment complexity of $O(|E|^{1.44})$ is presented in this section. This technique is a minor modification of a technique used by Muller and Preparata [7] for the restructuring of arithmetic expressions.

3.2.1 THE CONSTRUCTION

For this technique, the Boolean algebra of Boolean functions is used. (For details on Boolean algebras, see, for example, Harrison [3].)

Assume that an expression E has both regular literals and free variables $x_1, x_2, \ldots, x_e$. Then E can be represented in a Boolean algebra of Boolean functions and expressed in disjunctive normal form (DNF) with m_i products representing minterms and with switching functions as the discriminants. The expansion is of the form

$$E = \sum_{i=0}^{2^e-1} E_i m_i$$

where $E_0 = E(0,0,\ldots.0,0)$

$$E_1 = E(0,0,....0,1)$$
$$E_2 = E(0,0,....1,0)$$
$$\cdot$$
$$\cdot$$
$$E_{2^e-1} = E(1,1,....1,1)$$

(these E_i terms are the discriminants),

and where
$$m_0 = \bar{x}_e\bar{x}_{e-1}....\bar{x}_2\bar{x}_1$$
$$m_1 = \bar{x}_e\bar{x}_{e-1}....\bar{x}_2 x_1$$
$$\cdot$$
$$\cdot$$
$$m_{2^e-1} = x_e x_{e-1}....x_2 x_1$$

(these m_i terms are the minterms).

Note that in switching algebra the discriminants could only be the constants zero or one, but that in this case the discriminants are switching functions of the "regular" variables.

Define $t_2(E_i)$ as the minimum time required to compute both E_i and $\bar{E}_i$. Now define $t(E)$ as the maximum of all the times $t_2(E_i)$ over all i for the expression E.

LEMMA A:

Let A and B be two expressions with distinct variables and let t_A and t_B be the constructively achievable upper bounds to the computation times $t(A)$

and t(B) respectively. Then the upper bound

$$t(A \circ B) \leq 2+\max(t_A, t_B)$$

is constructively achievable.

PROOF:

Transform A and B into DNF around their free variables.

$$A = \sum_{i=0}^{2^a-1} A_i m_i$$

$$B = \sum_{i=0}^{2^b-1} B_i l_i$$

Now construct the DNF for $A \circ B$, the composition of A and B with respect to some free variable x_i in A.

Let $m_j = x_i k_h$.

Then in the DNF for $A \circ B$ there are terms as follows:

$$(A_j B_g) k_h l_g$$

and for the case in which, for example, $m_f = \bar{x}_i k_h$, there are terms as follows:

$$(A_f \bar{B}_g) k_h l_g.$$

Combining the two above terms gives

$$(A_j B_g + A_f \bar{B}_g) k_h l_g$$

as the only type of term in the DNF for $A \circ B$.

Note also that

$$(\bar{A}_j B_g + \bar{A}_f \bar{B}_g) k_h l_g$$

is the only type of term in the DNF for the complement of $A \circ B$. (Note that corresponding discriminants of $A \circ B$ and of the complement of $A \circ B$ are complements of each other.)

The discriminants are easily converted to NAND form.

Note that the time required to generate the discriminants of $A \circ B$ and $\overline{A \circ B}$ is $2+\max(t_A,t_B)$, which proves the lemma.

LEMMA B:

Let A and B be two expressions with distinct variables and let t_A and t_B be constructively achievable upper bounds to the computation times $t(A)$ and $t(B)$, respectively. Then the following upper bounds are achievable:

(i) $t(A+B) \leq 2+\max(t_A,t_B)$

(ii) $t(AB) \leq 2+\max(t_A,t_B)$

(iii) $t(A \oplus B) \leq 2+\max(t_A,t_B)$

PROOF:

Let DNF's be given for A and B which are written as in the proof of Lemma A. Then if m_j and l_i are minterms in A and B respectively, then

(i) There are only terms $(A_j+B_i)m_j l_i$ in the DNF

for A+B, and terms $(\bar{A}_j\bar{B}_i)m_jl_i$ in the DNF for the complement of A+B.

(ii) There are only terms $(A_jB_i)m_jl_i$ in the DNF for AB, and terms $(\bar{A}_j+\bar{B}_i)m_jl_i$ in the DNF for the complement of AB.

(iii) There are only terms $(A_j\bar{B}_i+\bar{A}_jB_i)m_jl_i$ in the DNF for A EXCLUSIVE-OR B, and terms $(A_jB_i+\bar{A}_j\bar{B}_i)m_jl_i$ in the DNF for the complement of A EXCLUSIVE-OR B.

Note that the discriminants can easily be implemented as NAND networks with a maximum of 2 levels thus the times required to compute the discriminants are as given in the lemma.

LEMMA C:

Let β be the positive root of $z^2+z-1 = 0$. ($\beta \simeq .6180$.)

then $t(E) \leq (-2/\log \beta\ ) \log |E| + S$

$\simeq\ 2.88 \log |E| + S$

is constructively achievable, where S is a constant.

PROOF:

Assume inductively that the bound applies for $|E| < n$, then, using Lemma 1, decompose E as

$A \circ (B \theta C)$ with

$|B| \leq |C| < \beta^2 n$

$|A| = |E| - |B| - |C| \leq n - \beta^2 n = \beta n$

since $\beta^2 + \beta - 1 = 0$, giving $\beta = 1 - \beta^2$

Thus both B and C can, by the inductive assumption, be constructed such that

$$t(C) \leq (-2/\log \beta)\log(\beta^2 n) + S$$
$$= (-2/\log \beta)\log n + S - 4$$

By Lemma B, $(B\theta C)$ can be formed in time

$$t(B\theta C) \leq 2 + \max(t(C), t(B))$$
$$= (-2/\log \beta)\log n + S - 2$$

By the inductive assumption, A can be constructed such that

$$t(A) \leq (-2/\log \beta)\log(\beta n) + S$$
$$= (-2/\log \beta)\log n + S - 2$$

By Lemma A,

$$t(A \circ (B\theta C)) \leq 2 + \max(t(A), t(B\theta C))$$
$$= (-2/\log \beta)\log n + S$$

Thus the induction extends to $|E| = n$.

Note that since Lemma C applies to any expression, if the expression has no free variables, the bound still applies. Thus again, the depth bound $2.88 \log n + O(1)$ is obtained for B2 to NAND conversion.

3.2.2 EQUIPMENT COMPLEXITY ANALYSIS

The equipment complexity analysis parallels the arguments in Muller and Preparata [7] for the calculation of $W_2(E)$ in their paper.

Let e, a, b, and c be the number of free variables in E, A, B, and C respectively. In forming $B\theta C$ there is a constant upper bound on the number of NAND gates required for each discriminant, so there are no more than $2^{b+c}K$ gates used in forming $B\theta C$. Similarly, in forming the composition of A with $B\theta C$ there are no more than $2^{e}K$ gates.

Thus the number of gates is bounded above by $K(2^{b+c}+2^{e})$, where $K > 0$ is some constant.

Let the number of gates used in the construction of E and $\bar{E}$ together be expressed as $Y(E)$.

Using Lemma C recursively gives

$$Y(E) \leq Y(A)+Y(B)+Y(C)+K(2^{b+c}+2^{e}).$$

Assume inductively that when $|E| < n$, then

$$Y(E) < K(2^{e}+1)(|E|^{w}-1)$$

where $w = (-1/\log \beta) \approx 1.44$.

Clearly, K can be chosen large enough to satisfy the Basis step.

Now let $|E| = n$

$$Y(E) < K(2^a)(|A|^w-1)+K(2^b+1)(|B|^w-1)$$
$$+K(2^c+1)(|C|^w-1)+K(2^{b+c}+2^e)$$

Note that Lemma 3 (section 1.5.2) applies for the case in which a, b, and c are treated as fixed and $|A|$, $|B|$, and $|C|$ are treated as variables.

Similarly, Lemma 3 applies for the case in which $|A|$, $|B|$, and $|C|$ are treated as fixed and a, b, and c are treated as variables.

By appropriate use of the above two types of application of Lemma 3, it can be seen that the upper bound of the right hand side of the above inequality occurs at

$$|A| = \beta n, \quad |B| = 1, \quad |C| = \beta^2 n-1$$
$$a = e+1, \quad b = 0, \quad \text{and} \quad c = 0, \quad \text{thus}$$
$$Y(E) < K((2^{e+1}+1)(\beta^w n^w-1)$$
$$+2((\beta^2 n-1)^w-1)+2^e+1)$$
$$< K((2^{e+1}+1)(\beta^w n^w-1)$$
$$+2(\beta^{2w} n^w-1)+2^e+1)$$

Now since $w = (-1/\log \beta)$, it can be seen that $\beta^w = 1/2$, thus

$$
\begin{aligned}
Y(E) &< K((2^{e+1}+1)(1/2\ n^w-1) \\
&\quad +2(1/4\ n^w-1)+2^e+1) \\
&= K(2^{e+1}(1/2\ n^w-1)+(1/2\ n^w-1) \\
&\quad +1/2\ n^w-2+2^e+1) \\
&= K(2^e(n^w-2)+n^w-2+2^e) \\
&= K((2^e+1)(n^w-2)+2^e) \\
&= K((2^e+1)(n^w-1)-1) \\
&< K(2^e+1)(n^w-1)
\end{aligned}
$$

Thus the inductive step is completed.

Note that if $p = 0$ (no free variables) then

$$Y(E) < 2K(|E|^{1.44}-1) < 2K|E|^{1.44}$$

3.3 3-INPUT NAND EQUIPMENT COMPLEXITY

The technique to obtain an upper bound on 3-input NAND equipment complexity of $O(|E|^{2.216})$ is presented here.

To reduce equipment complexity, the basic construction described in section 2.4 is modified to produce both E and $\overline{E}$ in the obvious manner similar to that used in section 3.1. In particular, the A_0 and Q_0 type terms are complemented to produce $\overline{E}$ and fan-out is used where possible to avoid duplication of terms.

ASSERTION:

Using the construction given in section 2.4, modified as described above, the equipment required is upper bounded by $O(|E|^{2.216})$.

PROOF:

Let the total number of gates used in the construction of E and $\bar{E}$ together be expressed as $Y(E)$.

Similarly, let the total number of gates used in the construction of E', E", $\bar{E}'$, and $\bar{E}''$ together be expressed as $Y_2(E)$.

Inductively assume that for $|E| < n$,

P1: $Y(E) \leq K(|E|^w - 1)$

P2: $Y_2(E) \leq K(|E|^w - 1)$

where $w = -\log((\sqrt{7}-2)/3) \simeq 2.216$
and K is a constant greater than or equal to 6.

Clearly, by the choice of a large enough value for K, the Basis step can be satisfied for the induction.

For the P1 constructions,

If $|B| \leq 1/4\ n$, (12 gates used in the combination network), then

$$Y(E) \leq 2Y_2(A) + Y(B) + Y_2(C) + 12$$
$$\leq K(2|A|^w + |B|^w + |C|^w) - K - 3K + 12$$

thus if $K \geq 4$,

$$Y(E) \leq Kn^W(2(|A|/n)^W+(|B|/n)^W+(|C|/n)^W)-K$$

Now, note that Lemma 3 applies, thus to maximize the right hand side of the inequality, only the extreme points of the values of $|A|/n$, $|B|/n$, and $|C|/n$ need be examined.

Similarly, if $|B| > 1/4\ n$, (18 gates used), then

$$Y(E) \leq Kn^W(2(|A|/n)^W+(|B|/n)^W+(|C|/n)^W)-K$$

where $K \geq 6$.

Similar arguments for P2 give

If $|Q| \leq 3/16\ n$, (38 gates used), then

$$Y(E) \leq Kn^W((|B|/n)^W+(|C|/n)^W+2(|Q|/n)^W +2(|R|/n)^W+4(|G|/n)^W)-K$$

where $K \geq 4 + 2/9$.

If $|Q| > 3/16\ n$, (42 gates used), then

$$Y(E) \leq Kn^W((|B|/n)^W+(|C|/n)^W+2(|Q|/n)^W +2(|R|/n)^W+4(|G|/n)^W)-K$$

where $K \geq 4 + 2/3$.

For the induction to hold, the term multiplied by Kn^W must be less than or equal to one in all four inequalities.

The limiting case is found to be the P2 construction where $|Q| \leq 3/16\ n$ and the extreme point is defined by

$$|B|/n = 0, \quad |C|/n = 1/4, \quad |Q|/n = 0,$$

$$|R|/n = 1/4, \quad \text{and} \quad |G|/n = 1/2.$$

Setting the term mentioned above to equal 1 gives

$$1 = (1/4)^w + 2(1/4)^w + 4(1/2)^w$$
$$= 3(1/2)^{2w} + 4(1/2)^w$$

Let $X = (1/2)^w$. Then

$1 = 3X^2 + 4X$, and the positive solution is

$X = (\sqrt{7}-2)/3$, thus

$(1/2)^w = (\sqrt{7}-2)/3$

giving $w = -\log((\sqrt{7}-2)/3) \simeq 2.216$

3.4 4-INPUT NAND EQUIPMENT COMPLEXITY

The technique to obtain an upper bound on 4-input NAND equipment complexity of $O(|E|^{1.832})$ is presented here.

To reduce equipment complexity, the basic construction described in section 2.5 is modified to produce both E and $\bar{E}$ in the obvious manner similar to that used in section 3.1. In particular, the A_0 and H_0 type terms are complemented to produce $\bar{E}$ and fan-out is used where possible to avoid duplication of terms.

ASSERTION:

Using the construction given in section 2.5, modified as described above, the equipment required is upper bounded by $O(|E|^{1.832})$.

PROOF:

Let the total number of gates used in the construction of E and $\bar{E}$ together be expressed as Y(E).

Let the total number of gates used in the construction of E', E", $\bar{E}$', and $\bar{E}$" together be expressed as $Y_2(E)$.

Let the total number of gates used in the construction of E"', E"", $\bar{E}$"', and $\bar{E}$"" together be expressed as $Y_3(E)$.

Inductively assume that for $|E| < n$,

P1: $Y(E) \leq K(|E|^w - 1)$

P2: $Y_2(E) \leq K(|E|^w - 1)$

P3: $Y_3(E) \leq K(|E|^w - 1)$

where $w = (\log(\sqrt{2}-1))/\log \beta \approx 1.832$,
and β is the positive root of $z^2+z-1 = 0$,
($\beta \approx .6180$.)
and K is a constant greater than or equal to 3 + 1/3.

Clearly, by the choice of a large enough value for K, the Basis step can be satisfied for the induction.

For the P1 and P3 constructions, the analysis is similar to the analysis portion of section 3.1 for the 2 input NAND equipment complexity, except that in the P1 case, the number of gates used in the construction is 10 and in the P3 case, the number of gates used in the construction is 8. Thus, K = 3 + 1/3.

For the P2 construction, call the B and C type terms formed after the decomposition of the A_0 terms B_A and C_A. Then

$$Y(E) \leq Y(B)+Y(C)+2Y(B_A)+2Y(C_A)+4Y(H)+24$$
$$= K(|B|^w+|C|^w+2|B_A|^w$$
$$+2|C_A|^w+4|H|^w-K-9K+24$$

thus if $K \geq 2 + 2/3$,

$$Y(E) \leq Kn^w((|B|/n)^w+(|C|/n)^w$$
$$+2(|B_A|/n)^w+2(|C_A|/n)^w)$$
$$+4(|H|/n)^w)-K$$

Now, note that Lemma 3 applies. Thus to maximize the right hand side of the inequality, only the extreme points of the values of $|B|/n$, $|C|/n$, $|B_A|/n$, $|C_A|/n$, and $|H|/n$ need be examined. For the induction to hold, the term multiplied by Kn^w must be less than or equal to one.

The maximum number of gates is found to be used at the extreme point defined by

$$|B|/n = 0, \quad |C|/n = \beta^2, \quad |B_A|/n = 0,$$
$$|C_A|/n = \beta^3, \quad \text{and} \quad |H|/n = \beta^2.$$

Setting the term mentioned above equal to 1 gives

$$1 = \beta^{2w}+2\beta^{3w}+4\beta^{2w}$$
$$1 = 5\beta^{2w}+2\beta^{3w}$$

Let $X = \beta^w$, then

$$1 = 5X^2+2X^3, \quad \text{giving}$$
$$2X^3+5X^2-1 = 0$$

Now note that in the P1 and P3 cases, the equation to

be solved was

$X^2+2X-1 = 0$, multiplication by 2X gives

$2X^3+4X^2-2X = 0$, and addition gives

$2X^3+5X^2-1 = 0$.

Thus, the roots of the P1 and P3 case equation are also roots of the P2 case equation.

Thus, the same solution holds for P2 and $w \simeq 1.832$ for $K \geq 3 + 1/3$.

CHAPTER 4

FORMULA CONVERSION FOR DEPTH REDUCTION WITH LINEAR EQUIPMENT

In this chapter, formula conversion for depth reduction will be described for the case in which the equipment growth is constrained to be linear. Depth results obtained are, for 2-input NAND conversion, $4 \log |E| + O(1)$; for 3-input NAND, $2.465 \log |E| + O(1)$; and for 4-input NAND, $2 \log |E| + O(1)$.

The approach used is similar to the approach used by Preparata and Muller [9],[10]. The key to linear equipment conversion is to insure that when a decomposition is performed, using lemma 1, succeeding decompositions are done such that succeeding choices for θ occur along the path from the original root to the original choice for θ. In graphic form this appears as shown in figure 15 for the original decomposition and in figure 16 for the next decomposition of A in figure 15.

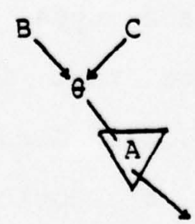

FIGURE 15 ORIGINAL DECOMPOSITION

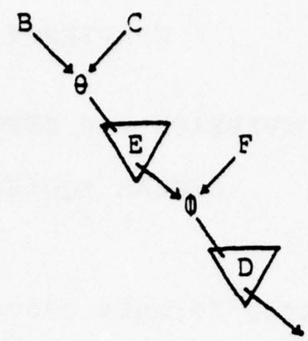

FIGURE 16 NEXT DECOMPOSITION

To accomplish the decomposition of A from figure 15, Lemma 2 is applied. The position in A at which ($B\theta C$) was attached is considered to be the free variable of A in the original expression. For some value, q, we decompose A such that $A = D \circ (F\Phi E)$, $|F\Phi E| \geq q$, E contains the free variable, and $|E| < q$.

Further decomposition of D and of E would be done similarly using Lemma 2. The next decompositions of B, C, and F would be done using Lemma 1, followed in turn by Lemma 2 type decompositions and Lemma 1 type decompositions of corresponding portions.

To perform Lemma 2 type decompositions, the general approach is to note that if x is a free variable in a Boolean expression G, then G can be restructured (by the Fundamental Theorem of Boolean Algebra) as $G = \bar{x}G_0+xG_1$. Decomposition of G_0 and G_1 is then accomplished.

Using Lemma 2 decompose $G = A \circ (B\theta C)$, where C contains the free variable, x, $|B\theta C| \geq q$, and $|C| < q$.

Now if θ = EXCLUSIVE-OR, then

$$G = (BC+\bar{B}\bar{C})A_0+(B\bar{C}+\bar{B}C)A_1$$

$$= (\bar{C}A_0+CA_1)\bar{B}+(CA_0+\bar{C}A_1)B$$

Now for C substitute $\bar{x}C_0+xC_1$ and expand about x to get $G = G_0\bar{x}+G_1x$ where

$$G_0 = (\bar{C}_0A_0+C_0A_1)\bar{B}+(C_0A_0+\bar{C}_0A_1)B$$

$$G_1 = (\bar{C}_1A_0+C_1A_1)\bar{B}+(C_1A_0+\bar{C}_1A_1)B$$

and similarly

$$\bar{G}_0 = (\bar{C}_0\bar{A}_0+C_0\bar{A}_1)\bar{B}+(C_0\bar{A}_0+\bar{C}_0\bar{A}_1)B$$

$$\bar{G}_1 = (\bar{C}_1\bar{A}_0+C_1\bar{A}_1)\bar{B}+(C_1\bar{A}_0+\bar{C}_1\bar{A}_1)B$$

This technique is applied recursively to obtain the A_0 and C_0 type terms.

4.1 B2 TO 2-INPUT NAND

The construction used to provide linear equipment growth for B2 to 2-input NAND conversion provides an upper bound on network depth of $4 \log |E| + O(1)$. The total equipment required is upper bounded by $K |E| - K$, where K is no greater than 18. The maximum gate fan-out used is 4 (not including input literal fan-out).

ASSERTION:

Any Boolean formula over the basis B2 can be restructured so that it describes a network of NAND gates with maximum fan-in of 2, with $O(|E|)$ equipment requirements, and with depth bounded from above by $4 \log |E| + O(1)$.

Proof:

Let E be a Boolean expression with $|E| \geq 1$.

Define $t_1(E)$ to be the minimum depth of any network realizing both expression E and $\bar{E}$ with total equipment $\leq K|E| - K$.

Define G to be a Boolean expression with a free variable x such that the expression can be restructured as $\bar{x}G_0+xG_1$. Let $t_2(G)$ be the minimum depth of any network which simultaneously realizes $G_0, \bar{G}_0, G_1$, and $\bar{G}_1$ with total equipment $\leq K|E| - K$.

Assume inductively:

P1: $t_1(E) \leq 4 \log |E| + S$

P2: $t_2(G) \leq 4 \log |G| + 2 + S$

Clearly, S can be chosen large enough to satisfy the Basis step for the induction.

The extension of P1 is provided by the following algorithm, where $n = |E|$ and $n > 2$. For $n \leq 2$, the construction is obvious and GATES(E) $\leq K|E|-K$ for $K = 18$.

STEP 1: Using Lemma 1, decompose E as

$A \circ (B \theta C)$ with

$|B| \leq |C| < 1/2\ n$

$|A| = |E| - |B| - |C| \leq n - 1/2\ n = 1/2\ n$

If θ is OR

$$E = \overline{(B+C)}A_0 + (B+C)A_1$$

$$= \bar{B}\bar{C}A_0 + (B+C)A_1$$

similarly

$$\bar{E} = \bar{B}\bar{C}\bar{A}_0 + (B+C)\bar{A}_1$$

As a NAND network this can be implemented as shown in figure 17.

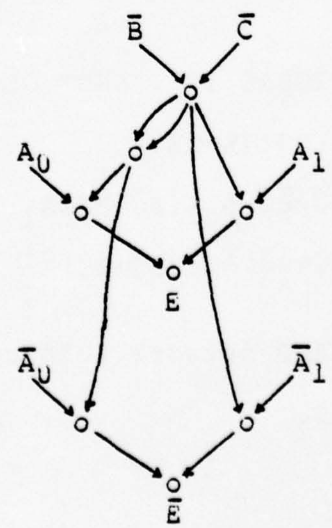

FIGURE 17 "OR" DECOMPOSITION

If θ is AND

$$E = \overline{(BC)}A_0+BCA_1$$
$$= (\bar{B}+\bar{C})A_0+BCA_1$$
$$\bar{E} = (\bar{B}+\bar{C})\bar{A}_0+BC\bar{A}_1$$

As a NAND network this can be implemented as shown in figure 18.

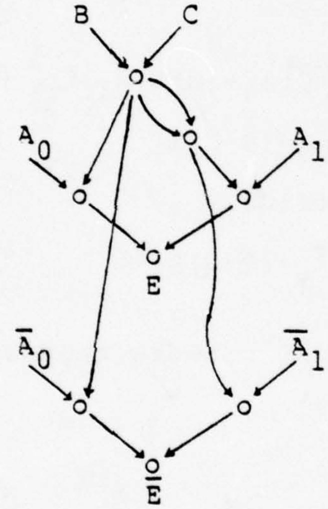

FIGURE 18 "AND" DECOMPOSITION

If θ is EXCLUSIVE-OR

$$E = (BC+\bar{B}\bar{C})A_0+(B\bar{C}+\bar{B}C)A_1$$
$$\bar{E} = (BC+\bar{B}\bar{C})\bar{A}_0+(B\bar{C}+\bar{B}C)\bar{A}_1$$

As a NAND network this can be implemented as shown in figure 19.

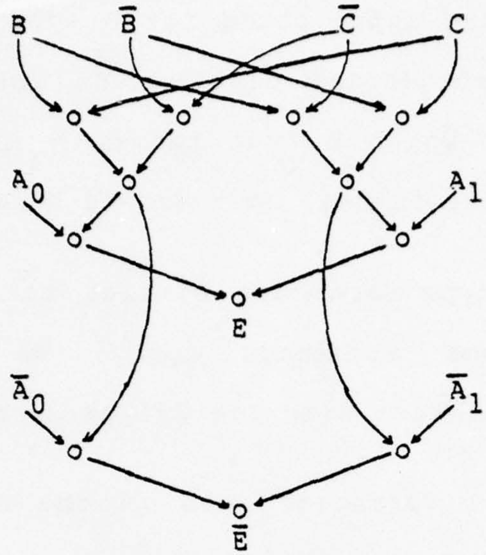

FIGURE 19 "EXCLUSIVE-OR" DECOMPOSITION

Thus in the worst case, B and C type terms are required in the construction by time T-4 and A_0 type terms are required by time T-2.

By P2 the A_0 type terms are of depth

$$t_2(A) \leq 4 \log(1/2\ n) + 2 + S$$
$$= 4 \log n + S - 2$$

Thus, the depth bound for A_0 type terms is 2 less than the hypothesized depth bound for E. But the latest time at which A_0 type terms in the expression must be available is T-2, so there is no depth contradiction.

By P1 the B type terms are of depth

$$t_1(B) \leq 4 \log(1/2\ n) + S$$
$$= 4 \log n + S - 4$$

Thus, the depth bound for B type terms is 4 less than the hypothesized depth bound for E. But the latest time at which B type terms in the expression must be available is T-4, so there is no depth contradiction.

The C type terms are similar to the B type terms and the same arguments apply, so there is no depth contradiction with the C type terms.

All variables used in the P1 decomposition fall into one of these three categories, so there is no depth contradiction in the P1 decomposition.

The combination equipment worst case is 12 gates.

$$\begin{aligned} GATES(E) &\le GATES(A)+GATES(B)+GATES(C)+12 \\ &\le K|A|-K+K|B|-K+K|C|-K+12 \\ &= K(|A|+|B|+|C|)-3K+12 \\ &= K|E|-K-2K+12 \end{aligned}$$

thus for $K = 18$,

$$GATES(E) \le K|E|-K$$

Thus there is no contradiction in the P1 equipment hypothesis.

The extension of P2 is provided by the following algorithm, where $n = |G|$ and $n > 2$. For $n \le 2$, the construction is obvious and $GATES(G) \le K|G|-K$ for $K = 18$.

STEP 1: Using Lemma 2, decompose G as

$A \circ (B\theta C)$ with

$|C| < 1/2\ n$,

C containing the free variable, x, and

$|A| = |E| - |B\theta C| \leq n - 1/2\ n = 1/2\ n.$

If θ is OR

$$G = \overline{(B+C)}A_0+(B+C)A_1$$
$$= \overline{B}\overline{C}A_0+(B+C)A_1$$
$$= (\overline{C}A_0+CA_1)\overline{B}+A_1B$$

Now for C substitute $\overline{x}C_0+xC_1$ and expand about x to get $G = G_0\overline{x}+G_1x$ where

$$G_0 = (\overline{C}_0A_0+C_0A_1)\overline{B}+A_1B$$
$$G_1 = (\overline{C}_1A_0+C_1A_1)\overline{B}+A_1B$$

and similarly

$$\overline{G}_0 = (\overline{C}_0\overline{A}_0+C_0\overline{A}_1)\overline{B}+\overline{A}_1B$$
$$\overline{G}_1 = (\overline{C}_1\overline{A}_0+C_1\overline{A}_1)\overline{B}+\overline{A}_1B$$

Constructing this network from NAND gates, in the obvious way, results in a network consisting of 22 gates.

If θ is AND

$$G = \overline{(BC)}A_0+BCA_1$$
$$= (\overline{B}+\overline{C})A_0+BCA_1$$
$$= A_0\overline{B}+(\overline{C}A_0+CA_1)B$$

Now for C substitute $\overline{x}C_0+xC_1$ and expand about x to get $G = G_0\overline{x}+G_1x$ where

$$G_0 = A_0\bar{B}+(\bar{C}_0A_0+C_0A_1)B$$

$$G_1 = A_0\bar{B}+(\bar{C}_1A_0+C_1A_1)B$$

and similarly

$$\bar{G}_0 = \bar{A}_0\bar{B}+(\bar{C}_0\bar{A}_0+C_0\bar{A}_1)B$$

$$\bar{G}_1 = \bar{A}_0\bar{B}+(\bar{C}_1\bar{A}_0+C_1\bar{A}_1)B$$

Constructing this network from NAND gates, in the obvious way, results in a network consisting of 22 gates.

If θ is EXCLUSIVE-OR

$$G = (BC+\bar{B}\bar{C})A_0+(B\bar{C}+\bar{B}C)A_1$$

$$= (\bar{C}A_0+CA_1)\bar{B}+(CA_0+\bar{C}A_1)B$$

Now for C substitute $\bar{x}C_0+xC_1$ and expand about x to get $G = G_0\bar{x}+G_1x$ where

$$G_0 = (\bar{C}_0A_0+C_0A_1)\bar{B}+(C_0A_0+\bar{C}_0A_1)B$$

$$G_1 = (\bar{C}_1A_0+C_1A_1)\bar{B}+(C_1A_0+\bar{C}_1A_1)B$$

and similarly

$$\bar{G}_0 = (\bar{C}_0\bar{A}_0+C_0\bar{A}_1)\bar{B}+(C_0\bar{A}_0+\bar{C}_0\bar{A}_1)B$$

$$\bar{G}_1 = (\bar{C}_1\bar{A}_0+C_1\bar{A}_1)\bar{B}+(C_1\bar{A}_0+\bar{C}_1\bar{A}_1)B$$

Constructing this network from NAND gates, in the obvious way, results in a network consisting of 36 gates.

Thus in the worst case in the constructions, A_0 type and C_0 type terms are required by time T-4, while B type terms are required by time T-2.

By P2 the A_0 type terms are of depth

$$t_2(A) \leq 4 \log(\ 1/2\ n) + 2 + S$$
$$= 4 \log n + 2 + S - 4$$

Thus, the depth bound for A_0 type terms is 4 less than the hypothesized depth bound for G. But the latest time at which A_0 type terms in the expression must be available is T-4, so there is no depth contradiction.

By P1 the B type terms are of depth

$$t_1(B) \leq 4 \log(\ n) + S$$
$$= 4 \log n + 2 + S - 2$$

Thus, the depth bound for B type terms is 2 less than the hypothesized depth bound for G. But the latest time at which B type terms in the expression must be available is T-2, so there is no depth contradiction.

The C_0 type terms are similar to the A_0 type terms and the same arguments apply, so there is no depth contradiction with the C_0 type terms.

All variables used in the P2 decomposition fall into one of these three categories, so there is no depth contradiction in the P2 decomposition.

The combination equipment worst case is 36 gates.

$$\text{GATES}(E) \leq \text{GATES}(A)+\text{GATES}(B)+\text{GATES}(C)+36$$
$$\leq K|A|-K+K|B|-K+K|C|-K+36$$

$$= K(|A|+|B|+|C|)-3K+36$$

$$= K|E|-K-2K+36$$

thus for $K = 18$,

$$GATES(E) \leq K|E|-K$$

Thus there is no contradiction in the P2 equipment hypothesis.

Thus the entire proof is complete.

For this construction, the implication graph is the complete graph on two nodes.

4.2 B2 TO 3-INPUT NAND

The construction used to provide linear equipment growth for B2 to 3-input NAND conversion provides an upper bound on network depth of $2.465 \log |E| + O(1)$. The total equipment required is upper bounded by $K\ |E| - K$, where K is no greater than 18. By analysis similar to that done for equipment requirements, it can be shown that the number of gate inputs used grows as $L\ |E| - L$, where L is no greater than 48, thus some inefficiency is evident since 18 gates would have 54 inputs. The maximum gate fan-out used is 4 (not including input literal fan-out).

with this particular construction, the results will be represented in terms of 3-input AND and OR gates which will alternate from the OR gate at the output, thus leaving a simple conversion to NAND gates for the end. Care must be taken to insure that the alternation between AND and OR gates is maintained. In some instances in this construction, an AND gate with a constant one input term and an OR gate with a constant zero input term is used to force the required alternation.

The construction and proof requires many pages so it is noted that succeeding material is not dependent upon a knowledge of the B2 to 3-input NAND case, thus allowing the reader without a need for the details to skip ahead.

ASSERTION:

Any Boolean formula over the basis B2 can be restructured so that it describes a network of NAND gates with maximum fan-in of 3, with $O(|E|)$ equipment requirements, and with depth bounded from above by $2.465 \log |E| + O(1)$.

Proof:

Let E be a Boolean expression with $|E| \geq 1$.

Define $t_1(E)$ to be the minimum depth of any network

realizing both expressions E and $\bar{E}$ with total equipment $\leq K\,|E| - K$, such that the output gates are <u>OR</u> gates and in all paths the gates alternate between AND and OR gates.

Define $t_2(E)$ to be the minimum depth of any network realizing both expressions E and $\bar{E}$ with total equipment $\leq K\,|E| - K$, such that the output gates are <u>AND</u> gates and in all paths the gates alternate between AND and OR gates.

Define $t_3(E)$ to be the minimum depth of any network with outputs E', E'', $\bar{E}'$, and $\bar{E}''$ with total equipment $\leq K\,|E| - K$, where

$$E = E'E'' \quad \text{and} \quad \bar{E} = \bar{E}'\bar{E}''.$$

The output gates must be <u>OR</u> gates and all paths must alternate between AND and OR gates.

Define $t_4(E)$ to be the minimum depth of any network with outputs E''', E'''', $\bar{E}'''$, and $\bar{E}''''$ with total equipment $\leq K\,|E| - K$, where

$$E = E''' + E'''' \quad \text{and} \quad \bar{E} = \bar{E}''' + \bar{E}''''.$$

The output gates must be <u>AND</u> gates and all paths must alternate between AND and OR gates.

Define G to be a Boolean expression with a free variable x such that the expression can be restructured as $\bar{x}G_0+xG_1$.

Let $t_5(G)$ be the minimum depth of any network which simultaneously realizes G_0, G_1 $\bar{G}_0$, and $\bar{G}_1$ with total equipment $\leq K\ |E| - K$, where

$$G = G_0\bar{x}+G_1x \qquad \text{and} \qquad \bar{G} = \bar{G}_0\bar{x}+\bar{G}_1x.$$

The output gates must be <u>OR</u> gates and all paths must alternate between AND and OR gates.

Let $t_6(G)$ be the minimum depth of any network which simultaneously realizes G_0, G_1 $\bar{G}_0$, and $\bar{G}_1$ with total equipment $\leq K\ |E| - K$, where

$$G = G_0\bar{x}+G_1x \qquad \text{and} \qquad \bar{G} = \bar{G}_0\bar{x}+\bar{G}_1x.$$

The output gates must be <u>AND</u> gates and all paths must alternate between AND and OR gates.

Let $t_7(G)$ be the minimum depth of any network which simultaneously realizes G_0', G_0'', G_1', G_1'', $\bar{G}_0'$, $\bar{G}_0''$, $\bar{G}_1'$, and $\bar{G}_1''$ with total equipment $\leq K\ |E| - K$, where

$$G_0 = G_0'G_0'' \quad , \quad \bar{G}_0 = \bar{G}_0'\bar{G}_0'',$$
$$G_1 = G_1'G_1'', \quad \text{and} \quad \bar{G}_1 = \bar{G}_1'\bar{G}_1''.$$

The output gates must be <u>OR</u> gates and all paths must alternate between AND and OR gates.

Let $t_8(G)$ be the minimum depth of any network which simultaneously realizes G_0''', G_0'''', G_1''', G_1'''', $\bar{G}_0'''$, $\bar{G}_0''''$, $\bar{G}_1'''$, and $\bar{G}_1''''$ with total equipment $\leq K\ |E| - K$, where

$$G_0 = G_0'''+G_0'''' \quad , \quad \bar{G}_0 = \bar{G}_0'''+\bar{G}_0'''',$$
$$G_1 = G_1'''+G_1'''', \quad \text{and} \quad \bar{G}_1 = \bar{G}_1'''+\bar{G}_1''''.$$

The output gates must be <u>AND</u> gates and all paths must alternate between AND and OR gates.

Define α as the positive root of $z^3+z^2-1=0$. Then α is approximately .7549 and $-1/\log \alpha$ is approximately 2.465.

Assume inductively:

P1: $t_1(E) \leq (-1/\log \alpha) \log |E| + S$

P2: $t_2(E) \leq (-1/\log \alpha) \log |E| + S$

P3: $t_3(E) \leq (-1/\log \alpha) \log |E| - 1 + S$

P4: $t_4(E) \leq (-1/\log \alpha) \log |E| - 1 + S$

P5: $t_5(G) \leq (-1/\log \alpha) \log |G| + 1 + S$

P6: $t_6(G) \leq (-1/\log \alpha) \log |G| + 1 + S$

P7: $t_7(G) \leq (-1/\log \alpha) \log |G| + S$

P8: $t_8(G) \leq (-1/\log \alpha) \log |G| + S$

Clearly, S can be chosen large enough to satisfy the Basis step for the induction.

The extension of P1 is provided by the following algorithm, where $n = |E|$ and $n > 2$. For $n \leq 2$, the construction is obvious and $\text{GATES}(E) \leq K|E|-K$ for $K = 18$.

STEP 1: Using Lemma 1, decompose E as

$A \circ (B\theta C)$ with

$|B| \leq |C| < \alpha^3 n$

$|A| = |E| - |B| - |C| \leq n - \alpha^3 n = \alpha^2 n$

since $\alpha^3 + \alpha^2 - 1 = 0$ giving $\alpha^2 = 1 - \alpha^3$

STEP 2: If $|B| \leq \alpha^4 n$

then for θ = EXCLUSIVE-OR

$$E = (BC+\bar{B}\bar{C})A_0 + (B\bar{C}+\bar{B}C)A_1$$

$$= (BC'C''+\bar{B}\bar{C}'\bar{C}'')A_0'A_0'' + (B\bar{C}'\bar{C}''+\bar{B}C'C'')A_1'A_1''$$

similarly

$$\bar{E} = (BC'C''+\bar{B}\bar{C}'\bar{C}'')\bar{A}_0'\bar{A}_0'' + (B\bar{C}'\bar{C}''+\bar{B}C'C'')\bar{A}_1'\bar{A}_1''$$

This portion of the network consists of 12 gates and uses 32 gate inputs.

By P7 the A_0' type terms are of depth

$$t_7(A) \leq (-1/\log \alpha) \log(\alpha^2 n) + S$$

$$= (-1/\log \alpha) \log n + S - 2$$

Thus, the depth bound for A_0' type terms is 2 less than the hypothesized depth bound for E. But the latest time at which A_0' type terms in the expression must be available is T-2, so there is no depth contradiction.

By P1 the B type terms are of depth

$$t_1(B) \leq (-1/\log \alpha) \log(\alpha^4 n) + S$$

$$= (-1/\log \alpha) \log n + S - 4$$

Thus, the depth bound for B type terms is 4 less than the hypothesized depth bound for E. But the latest time at which B type terms in the expression must be

available is T-4, so there is no depth contradiction.

By P3 the C' type terms are of depth

$$t_3(C) \leq (-1/\log\alpha)\ \log(\alpha^3 n) - 1 + S$$
$$= (-1/\log\alpha)\ \log n + S - 4$$

Thus, the depth bound for C' type terms is 4 less than the hypothesized depth bound for E. But the latest time at which C' type terms in the expression must be available is T-4, so there is no depth contradiction.

All variables used in the step 2 EXCLUSIVE-OR decomposition fall into one of these three categories, so there is no depth contradiction in the step 2 EXCLUSIVE-OR decomposition.

then for θ = AND

$$E = \overline{(BC)}A_0 + (BC)A_1$$
$$= (\bar{B}+\bar{C})A_0 + (BC)A_1$$
$$= (\bar{B}1+\bar{C}'\bar{C}'')A_0'A_0'' + (BC'C''+0)A_1'A_1''$$

similarly

$$\bar{E} = (\bar{B}1+\bar{C}'\bar{C}'')\bar{A}_0'\bar{A}_0'' + (BC'C''+0)\bar{A}_1'\bar{A}_1''$$

All terms are satisfactory, based upon the previous arguments. Thus there is no depth contradiction in the step 2 AND decomposition.

then for θ = OR

$$E = \overline{(B+C)}A_0+(B+C)A_1$$
$$= (\bar{B}\bar{C})A_0+(B+C)A_1$$
$$= (\bar{B}\bar{C}'\bar{C}''+0)A_0'A_0''+(B1+C'C'')A_1'A_1''$$

similarly

$$\bar{E} = (\bar{B}\bar{C}'\bar{C}''+0)\bar{A}_0'\bar{A}_0''+(B1+C'C'')\bar{A}_1'\bar{A}_1''$$

All term types used have been evaluated in other parts of step 2 and been found satisfactory, so there is no depth contradiction in step 2.

STEP 3: If $|B| > \alpha^4 n$

then $|A| = |E|-|B|-|C| < (1-2\alpha^4)n$

but $\alpha > 1/\sqrt{2}$, thus $1-2\alpha^4 < 1-\alpha^2$

and since $\alpha^3+\alpha^2-1 = 0$, then $1-\alpha^2 = \alpha^3$

giving $|A| < \alpha^3 n$

then for θ = EXCLUSIVE-OR

$$E = (B+\bar{C})(\bar{B}+C)A_0+(B+C)(\bar{B}+\bar{C})A_1$$
$$\bar{E} = (B+\bar{C})(\bar{B}+C)\bar{A}_0+(B+C)(\bar{B}+\bar{C})\bar{A}_1$$

This portion of the network consists of 10 gates and uses 24 gate inputs.

By P5 the A_0 type terms are of depth

$$t_5(A) \le (-1/\log\alpha)\log(\alpha^3 n) + 1 + S$$
$$=(-1/\log\alpha)\log n + S - 2$$

Thus, the depth bound for A_0 type terms is 2 less than the hypothesized depth bound for E. But the latest time at which A_0 type terms in the expression must be available is T-2, so there is no depth contradiction.

By P2 the B type terms are of depth

$$t_2(B) \le (-1/\log\alpha)\ \log(\alpha^3 n) + S$$
$$= (-1/\log\alpha)\ \log n + S - 3$$

Thus, the depth bound for B type terms is 3 less than the hypothesized depth bound for E. But the latest time at which B type terms in the expression must be available is T-3, so there is no depth contradiction.

The C type terms are similar to the B type terms and the same arguments apply, so there is no depth contradiction with the C type terms.

All variables used in the step 3 EXCLUSIVE-OR decomposition fall into one of these three categories, so there is no depth contradiction in the step 3 EXCLUSIVE-OR decomposition.

then for θ = AND

$$E = \overline{(BC)}A_0 + (BC)A_1$$
$$= (\overline{B}+\overline{C})A_0 + (B+0)(C+0)A_1$$
$$\overline{E} = (\overline{B}+\overline{C})\overline{A}_0 + (B+0)(C+0)\overline{A}_1$$

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All terms are satisfactory, based upon the previous arguments. Thus there is no depth contradiction in the step 3 AND decomposition.

then for θ = OR

$$E = \overline{(B+C)}A_0+(B+C)A_1$$
$$= (\bar{B}+0)(\bar{C}+0)A_0+(B+C)A_1$$
$$\bar{E} = (\bar{B}+0)(\bar{C}+0)\bar{A}_0+(B+C)\bar{A}_1$$

All term types used have been evaluated in other parts of step 3 and been found satisfactory, so there is no depth contradiction in step 3. Thus, there is no depth contradiction in the P1 decomposition.

The combination equipment worst case is 12 gates.

$$\begin{aligned} \text{GATES}(E) &\leq \text{GATES}(A)+\text{GATES}(B)+\text{GATES}(C)+12 \\ &\leq K|A|-K+K|B|-K+K|C|-K+12 \\ &= K(|A|+|B|+|C|)-3K+12 \\ &= K|E|-K-2K+12 \end{aligned}$$

thus for $K = 18$,

$$\text{GATES}(E) \leq K|E|-K$$

Thus there is no contradiction in the P1 equipment hypothesis.

The extension of P2 is provided by the following algorithm, where $n = |E|$ and $n > 2$. In essence P2 is the dual of P1. For $n \leq 2$ the construction is obvious and $GATES(E) \leq K|E|-K$ for $K = 18$.

STEP 1: Using Lemma 1, decompose E as

$$A \circ (B \theta C) \text{ with}$$

$$|B| \leq |C| < \alpha^3 n$$

$$|A| = |E| - |B| - |C| \leq n - \alpha^3 n = \alpha^2 n$$

$$\text{since} \quad \alpha^3 + \alpha^2 - 1 = 0 \quad \text{giving} \quad \alpha^2 = 1 - \alpha^3$$

STEP 2: If $|B| \leq \alpha^4 n$

then for θ = EXCLUSIVE-OR

$$E = ((B+C)(\bar{B}+\bar{C})+A_0)((B+\bar{C})(\bar{B}+C)+A_1)$$

$$= ((B+C"'+C"")(\bar{B}+\bar{C}"'+\bar{C}"")+A_0"'+A_0"")$$

$$((B+\bar{C}"'+\bar{C}"")(\bar{B}+C"'+C"")+A_1"'+A_1"")$$

similarly

$$\bar{E} = ((B+C"'+C"")(\bar{B}+\bar{C}"'+\bar{C}"")+\bar{A}_0"'+\bar{A}_0"")$$

$$((B+\bar{C}"'+\bar{C}"")(\bar{B}+C"'+C"")+\bar{A}_1"'+\bar{A}_1"")$$

This portion of the network consists of 12 gates and uses 32 gate inputs.

By P8 the $A_0"'$ type terms are of depth

$$t_8(A) \leq (-1/\log\alpha)\log(\alpha^2 n) + S$$

$$= (-1/\log\alpha)\log n + S - 2$$

Thus, the depth bound for $A_0"'$ type terms is 2 less than the hypothesized depth bound for E. But the

latest time at which A_0''' type terms in the expression must be available is T-2, so there is no depth contradiction.

By P2 the B type terms are of depth

$$t_2(B) \leq (-1/\log\alpha)\log(\alpha^4 n) + S$$
$$= (-1/\log\alpha)\log n + S - 4$$

Thus, the depth bound for B type terms is 4 less than the hypothesized depth bound for E. But the latest time at which B type terms in the expression must be available is T-4, so there is no depth contradiction.

By P4 the C"' type terms are of depth

$$t_4(C) \leq (-1/\log\alpha)\log(\alpha^3 n) - 1 + S$$
$$= (-1/\log\alpha)\log n + S - 4$$

Thus, the depth bound for C"' type terms is 4 less than the hypothesized depth bound for E. But the latest time at which C"' type terms in the expression must be available is T-4, so there is no depth contradiction.

All variables used in the step 2 EXCLUSIVE-OR decomposition fall into one of these three categories, so there is no depth contradiction in the step 2 EXCLUSIVE-OR decomposition.

then for θ = AND

$$E = (BC+A_0)(\overline{(BC)}+A_1)$$

$$= (BC+A_0)((\bar{B}+\bar{C})+A_1)$$

$$= ((B+0)(C''{}'+C'''')+A_0''{}'+A_0'''')$$

$$((\bar{B}+\bar{C}''{}'+\bar{C}'''')1+A_1''{}'+A_1'''')$$

similarly

$$\bar{E} = ((B+0)(C''{}'+C'''')+\bar{A}_0''{}'+\bar{A}_0'''')$$

$$((\bar{B}+\bar{C}''{}'+\bar{C}'''')1+\bar{A}_1''{}'+\bar{A}_1'''')$$

All terms are satisfactory, based upon the previous arguments. Thus there is no depth contradiction in the step 2 AND decomposition.

then for θ = OR

$$E = ((B+C)+A_0)(\overline{(B+C)}+A_1)$$

$$= ((B+C)+A_0)(\bar{B}\bar{C}+A_1)$$

$$= (B+C''{}'+C'''')1+A_0''{}'+A_0'''')$$

$$((\bar{B}+0)(\bar{C}''{}'+\bar{C}'''')+A_1''{}'+A_1'''')$$

similarly

$$\bar{E} = (B+C''{}'+C'''')1+\bar{A}_0''{}'+\bar{A}_0'''')$$

$$((\bar{B}+0)(\bar{C}''{}'+\bar{C}'''')+\bar{A}_1''{}'+\bar{A}_1'''')$$

All term types used have been evaluated in other parts of step 2 and been found satisfactory, so there is no depth contradiction in step 2.

STEP 3: If $|B| > \alpha^4 n$

then $|A| = |E|-|B|-|C| < (1-2\,\alpha^4)n$

but $\alpha > 1/\sqrt{2}$, thus $1-2\,\alpha^4 < 1-\alpha^2$

and since $\alpha^3+\alpha^2-1 = 0$, then $1-\alpha^2 = \alpha^3$

giving $|A| < \alpha^3 n$

then for θ = EXCLUSIVE-OR

$$E = (BC+\bar{B}\bar{C}+A_0)(B\bar{C}+\bar{B}C+A_1)$$

$$\bar{E} = (BC+\bar{B}\bar{C}+\bar{A}_0)(B\bar{C}+\bar{B}C+\bar{A}_1)$$

This portion of the network consists of 10 gates and uses 24 gate inputs.

By P6 the A_0 type terms are of depth

$$t_6(A) \leq (-1/\log\alpha)\log(\alpha^3 n) + 1 + S$$
$$= (-1/\log\alpha)\log n + S - 2$$

Thus, the depth bound for A_0 type terms is 2 less than the hypothesized depth bound for E. But the latest time at which A_0 type terms in the expression must be available is T-2, so there is no depth contradiction.

By P1 the B type terms are of depth

$$t_1(B) \leq (-1/\log\alpha)\log(\alpha^3 n) + S$$
$$= (-1/\log\alpha)\log n + S - 3$$

Thus, the depth bound for B type terms is 3 less than the hypothesized depth bound for E. But the latest time at which B type terms in the expression must be available is T-3, so there is no depth contradiction.

The C type terms are similar to the B type terms and the same arguments apply, so there is no depth contradiction with the C type terms.

All variables used in the step 3 EXCLUSIVE-OR decomposition fall into one of these three categories, so there is no depth contradiction in the step 3 EXCLUSIVE-OR decomposition.

then for θ = AND

$$E = (BC+A_0)(\overline{BC}+A_1)$$
$$= (BC+A_0)(\bar{B}+\bar{C}+A_1)$$
$$= (BC+A_0)(\bar{B}1+\bar{C}1+A_1)$$
$$\bar{E} = (BC+\bar{A}_0)(\bar{B}1+\bar{C}1+\bar{A}_1)$$

All terms are satisfactory, based upon the previous arguments. Thus there is no depth contradiction in the step 3 AND decomposition.

then for θ = OR

$$E = (B+C+A_0)(\overline{(B+C)}+A_1)$$
$$= (B+C+A_0)(\bar{B}\bar{C}+A_1)$$
$$= (B1+C1+A_0)(\bar{B}\bar{C}+A_1)$$
$$\bar{E} = (B1+C1+\bar{A}_0)(\bar{B}\bar{C}+\bar{A}_1)$$

All term types used have been evaluated in other parts of step 3 and been found satisfactory, so there is no depth contradiction in step 3. Thus, there is no depth contradiction in the P2 decomposition.

The combination equipment worst case is 12 gates..

$$\begin{aligned} GATES(E) &\leq GATES(A)+GATES(B)+GATES(C)+12 \\ &\leq K|A|-K+K|B|-K+K|C|-K+12 \\ &= K(|A|+|B|+|C|)-3K+12 \\ &= K|E|-K-2K+12 \end{aligned}$$

thus for $K = 18$,

$$GATES(E) \leq K|E|-K$$

Thus there is no contradiction in the P2 equipment hypothesis.

The extension of P3 is provided by the following argument:

The arguments and construction for P3 are essentially the same as for P2, except that the two terms at the output of the P2 constructions are separated and treated independently as E' and E" etc.

The extension of P4 is provided by the following argument:

The arguments and construction for P4 are essentially the same as for P1, except that the two terms at the output of the P1 constructions are separated and treated independently as E"' and E"" etc.

The extension of P5 is provided by the following algorithm, where $n = |G|$ and $n > 1$. For $n = 1$, the construction is obvious and $\text{GATES}(G) \leq K|G|-K$ for $K = 18$.

STEP 1: Using Lemma 2, decompose G as

$A \circ (B \theta C)$ with

$|C| < \alpha n$,

C containing the free variable, x, and

$|A| = |E| - |B\theta C| \leq n - \alpha n = (1-\alpha)n.$

recall that $\alpha^3 + \alpha^2 - 1 = 0$

thus $\alpha^4 + \alpha^3 - \alpha = 0$

and, by subtracting, $\alpha^4 - \alpha^2 - \alpha + 1 = 0$

thus $1-\alpha = \alpha^2 - \alpha^4 = \alpha^2(1-\alpha^2) = \alpha^5$

but $|A| \leq (1-\alpha)n$

thus $|A| \leq \alpha^5 n$

STEP 2: If $|B| \leq \alpha^2 n$

then for θ = EXCLUSIVE-OR

$$G = (BC+\bar{B}\bar{C})A_0 + (B\bar{C}+\bar{B}C)A_1$$

$$= (\bar{B}A_0 + BA_1)\bar{C} + (BA_0 + \bar{B}A_1)C$$

$$= (\bar{B}'\bar{B}''A_0+B'B''A_1)\bar{C}'\bar{C}''+(B'B''A_0+\bar{B}'\bar{B}''A_1)C'C''$$

Now for C substitute $\bar{x}C_0+xC_1$ and expand about x to get $G = G_0\bar{x}+G_1x$ where

$$G_0 = (\bar{B}'\bar{B}''A_0+B'B''A_1)\bar{C}_0'\bar{C}_0''+(B'B''A_0+\bar{B}'\bar{B}''A_1)C_0'C_0''$$

$$G_1 = (\bar{B}'\bar{B}''A_0+B'B''A_1)\bar{C}_1'\bar{C}_1''+(B'B''A_0+\bar{B}'\bar{B}''A_1)C_1'C_1''$$

similarly

$$\bar{G}_0 = (\bar{B}'\bar{B}''\bar{A}_0+B'B''\bar{A}_1)\bar{C}_0'\bar{C}_0''+(B'B''\bar{A}_0+\bar{B}'\bar{B}''\bar{A}_1)C_0'C_0''$$

$$\bar{G}_1 = (\bar{B}'\bar{B}''\bar{A}_0+B'B''\bar{A}_1)\bar{C}_1'\bar{C}_1''+(B'B''\bar{A}_0+\bar{B}'\bar{B}''\bar{A}_1)C_1'C_1''$$

This portion of the network consists of 24 gates and uses 64 gate inputs.

By P5 the A_0 type terms are of depth

$$t_5(A) \leq (-1/\log \alpha) \log(\alpha^5 n) + 1 + S$$
$$=(-1/\log \alpha) \log n + 1 + S - 5$$

Thus, the depth bound for A_0 type terms is 5 less than the hypothesized depth bound for G_0. But the latest time at which A_0 type terms in the expression must be available is T-4, so there is no depth contradiction.

By P3 the B' type terms are of depth

$$t_3(B) \leq (-1/\log \alpha) \log(\alpha^2 n) - 1 + S$$
$$=(-1/\log \alpha) \log n + 1 + S - 4$$

Thus, the depth bound for B' type terms is 4 less than the hypothesized depth bound for G_0. But the latest time at which B' type terms in the expression must be

available is T-4, so there is no depth contradiction.

By P7 the C_0' type terms are of depth

$$t_7(C) \leq (-1/\log\alpha)\log(\alpha n) + S$$
$$= (-1/\log\alpha)\log n + 1 + S - 2$$

Thus, the depth bound for C_0' type terms is 2 less than the hypothesized depth bound for G_0. But the latest time at which C_0' type terms in the expression must be available is T-2, so there is no depth contradiction.

All variables used in the step 2 EXCLUSIVE-OR decomposition fall into one of these three categories, so there is no depth contradiction in the step 2 EXCLUSIVE-OR decomposition.

then for θ = AND

$$G = (\overline{BC})A_0 + BCA_1$$
$$= (\overline{B}+\overline{C})A_0 + BCA_1$$
$$= A_0\overline{C} + (\overline{B}A_0 + BA_1)C$$
$$= A_0\overline{C}'\overline{C}'' + (\overline{B}'\overline{B}''A_0 + B'B''A_1)C'C''$$

Now for C substitute $\overline{x}C_0 + xC_1$ and expand about x to get $G = G_0\overline{x} + G_1 x$ where

$$G_0 = A_0\overline{C}_0'\overline{C}_0'' + (\overline{B}'\overline{B}''A_0 + B'B''A_1)C_0'C_0''$$
$$G_1 = A_0\overline{C}_1'\overline{C}_1'' + (\overline{B}'\overline{B}''A_0 + B'B''A_1)C_1'C_1''$$

similarly

$$\overline{G}_0 = \overline{A}_0\overline{C}_0'\overline{C}_0'' + (\overline{B}'\overline{B}''\overline{A}_0 + B'B''\overline{A}_1)C_0'C_0''$$

$$\overline{G}_1 = \overline{A}_0\overline{C}_1'\overline{C}_1'' + (\overline{B}'\overline{B}''\overline{A}_0 + B'B''\overline{A}_1)C_1'C_1''$$

All terms are satisfactory, based upon the previous arguments. Thus there is no depth contradiction in the step 2 AND decomposition.

then for θ = OR

$$G = \overline{(B+C)}A_0 + (B+C)A_1$$
$$= \overline{B}\overline{C}A_0 + (B+C)A_1$$
$$= (\overline{B}A_0 + BA_1)\overline{C} + A_1C$$
$$= (\overline{B}'\overline{B}''A_0 + B'B''A_1)\overline{C}'\overline{C}'' + A_1C'C''$$

Now for C substitute $\overline{x}C_0 + xC_1$ and expand about x to get $G = G_0\overline{x} + G_1x$ where

$$G_0 = (\overline{B}'\overline{B}''A_0 + B'B''A_1)\overline{C}_0'\overline{C}_0'' + A_1C_0'C_0''$$
$$G_1 = (\overline{B}'\overline{B}''A_0 + B'B''A_1)\overline{C}_1'\overline{C}_1'' + A_1C_1'C_1''$$

similarly

$$\overline{G}_0 = (\overline{B}'\overline{B}''\overline{A}_0 + B'B''\overline{A}_1)\overline{C}_0'\overline{C}_0'' + \overline{A}_1C_0'C_0''$$
$$\overline{G}_1 = (\overline{B}'\overline{B}''\overline{A}_0 + B'B''\overline{A}_1)\overline{C}_1'\overline{C}_1'' + \overline{A}_1C_1'C_1''$$

All term types used have been evaluated in other parts of step 2 and been found satisfactory, so there is no depth contradiction in step 2.

STEP 3: If $|B| > \alpha^2 n$

Then $|C| < n - \alpha^2 n = (1 - \alpha^2)n = \alpha^3 n$

since $\alpha^3 + \alpha^2 - 1 = 0$

then for θ = EXCLUSIVE-OR

$$G = (BC+\bar{B}\bar{C})A_0+(B\bar{C}+\bar{B}C)A_1$$
$$= (\bar{C}A_0+CA_1)\bar{B}+(CA_0+\bar{C}A_1)B$$

By using arguments similar to those in step 2 for EXCLUSIVE-OR, the following constructions are obtained:

$$G_0 = (\bar{C}'_0\bar{C}''_0A_0+C'_0C''_0A_1)\bar{B}'\bar{B}''$$
$$+(C'_0C''_0A_0+\bar{C}'_0\bar{C}''_0A_1)B'B''$$
$$G_1 = (\bar{C}'_1\bar{C}''_1A_0+C'_1C''_1A_1)\bar{B}'\bar{B}''$$
$$+(C'_1C''_1A_0+\bar{C}'_1\bar{C}''_1A_1)B'B''$$
$$\bar{G}_0 = (\bar{C}'_0\bar{C}''_0\bar{A}_0+C'_0C''_0\bar{A}_1)\bar{B}'\bar{B}''$$
$$+(C'_0C''_0\bar{A}_0+\bar{C}'_0\bar{C}''_0\bar{A}_1)B'B''$$
$$\bar{G}_1 = (\bar{C}'_1\bar{C}''_1\bar{A}_0+C'_1C''_1\bar{A}_1)\bar{B}'\bar{B}''$$
$$+(C'_1C''_1\bar{A}_0+\bar{C}'_1\bar{C}''_1\bar{A}_1)B'B''$$

This portion of the network consists of 36 gates and uses 96 gate inputs.

The argument for A_0 type terms is identical to the argument for A_0 type terms in step 2, so there is no depth contradiction with the A_0 type terms.

By P3 the B' type terms are of depth

$$t_3(B) \leq (-1/\log\alpha)\log(n) - 1 + S$$
$$=(-1/\log\alpha)\log n + 1 + S - 2$$

Thus, the depth bound for B' type terms is 2 less than the hypothesized depth bound for G_0. But the latest time at which B' type terms in the expression must be

available is T-2, so there is no depth contradiction.

By P7 the C_0' type terms are of depth

$$t_7(C) \leq (-1/\log \alpha) \log(\alpha^3 n) + S$$
$$= (-1/\log \alpha) \log n + 1 + S - 4$$

Thus, the depth bound for C_0' type terms is 4 less than the hypothesized depth bound for G_0. But the latest time at which C_0' type terms in the expression must be available is T-4, so there is no depth contradiction.

All variables used in the step 3 EXCLUSIVE-OR decomposition fall into one of these three categories, so there is no depth contradiction in the step 3 EXCLUSIVE-OR decomposition.

then for θ = AND

$$G = \overline{(BC)}A_0 + BCA_1$$
$$= (\bar{B}+\bar{C})A_0 + BCA_1$$
$$= A_0\bar{B} + (\bar{C}A_0 + CA_1)B$$

By using arguments similar to those in step 2 for AND, the following constructions are obtained:

$$G_0 = A_0\bar{B}'\bar{B}'' + (\bar{C}_0'\bar{C}_0''A_0 + C_0'C_0''A_1)B'B''$$
$$G_1 = A_0\bar{B}'\bar{B}'' + (\bar{C}_1'\bar{C}_1''A_0 + C_1'C_1''A_1)B'B''$$
$$\bar{G}_0 = \bar{A}_0\bar{B}'\bar{B}'' + (\bar{C}_0'\bar{C}_0''\bar{A}_0 + C_0'C_0''\bar{A}_1)B'B''$$
$$\bar{G}_1 = \bar{A}_0\bar{B}'\bar{B}'' + (\bar{C}_1'\bar{C}_1''\bar{A}_0 + C_1'C_1''\bar{A}_1)B'B''$$

All terms are satisfactory, based upon the previous arguments.

then for θ = OR

$$
\begin{aligned}
G &= \overline{(B+C)}A_0+(B+C)A_1 \\
&= \bar{B}\bar{C}A_0+(B+C)A_1 \\
&= (\bar{C}A_0+CA_1)\bar{B}+A_1B
\end{aligned}
$$

By using arguments similar to those in step 2 for OR, the following constructions are obtained:

$$
\begin{aligned}
G_0 &= (\bar{C}'_0\bar{C}''_0A_0+C'_0C''_0A_1)\bar{B}'\bar{B}''+A_1B'B'' \\
G_1 &= (\bar{C}'_1\bar{C}''_1A_0+C'_1C''_1A_1)\bar{B}'\bar{B}''+A_1B'B'' \\
\bar{G}_0 &= (\bar{C}'_0\bar{C}''_0\bar{A}_0+C'_0C''_0\bar{A}_1)\bar{B}'\bar{B}''+\bar{A}_1B'B'' \\
\bar{G}_1 &= (\bar{C}'_1\bar{C}''_1\bar{A}_0+C'_1C''_1\bar{A}_1)\bar{B}'\bar{B}''+\bar{A}_1B'B''
\end{aligned}
$$

All term types used have been evaluated in other parts of step 3 and been found satisfactory, so there is no depth contradiction in step 3. Thus, there is no depth contradiction in the P5 decomposition.

The combination equipment worst case is 36 gates.

$$
\begin{aligned}
\text{GATES}(E) &\le \text{GATES}(A)+\text{GATES}(B)+\text{GATES}(C)+36 \\
&\le K|A|-K+K|B|-K+K|C|-K+36 \\
&= K(|A|+|B|+|C|)-3K+36 \\
&= K|E|-K-2K+36
\end{aligned}
$$

thus for $K = 18$,

$$\text{GATES}(E) \le K|E|-K$$

Thus there is no contradiction in the P5 equipment hypothesis.

The extension of P6 is provided by the following algorithm, where $n = |G|$ and $n > 1$. In essence, P6 is the dual of P5. For $n = 1$, the construction is obvious and $\mathrm{GATES}(G) \leq K|E|-K$ for $K = 18$.

STEP 1: Using Lemma 2, decompose G as

$A \circ (B\theta C)$ with

$|C| < \alpha n$,

C containing the free variable, x, and

$|A| = |E| - |B\theta C| \leq n - \alpha n = (1-\alpha)n.$

recall that $\alpha^3 + \alpha^2 - 1 = 0$

thus $\alpha^4 + \alpha^3 - \alpha = 0$

and, by subtracting, $\alpha^4 - \alpha^2 - \alpha + 1 = 0$

thus $1-\alpha = \alpha^2 - \alpha^4 = \alpha^2(1-\alpha^2) = \alpha^5$

but $|A| \leq (1-\alpha)n$

thus $|A| \leq \alpha^5 n$

STEP 2: If $|B| \leq \alpha^2 n$

then for θ = EXCLUSIVE-OR

$$\overline{G} = (BC+\overline{B}\overline{C})\overline{A}_0 + (B\overline{C}+\overline{B}C)\overline{A}_1$$

$$= (\overline{B}\overline{A}_0 + B\overline{A}_1)\overline{C} + (B\overline{A}_0 + \overline{B}\overline{A}_1)C$$

$$G = ((B+A_0)(\overline{B}+A_1)+C)((\overline{B}+A_0)(B+A_1)+\overline{C})$$

$$G = ((B''' + B'''' + A_0)(\overline{B}''' + \overline{B}'''' + A_1) + C''' + C'''')$$

$$((\overline{B}''' + \overline{B}'''' + A_0)(B''' + B'''' + A_1) + \overline{C}''' + \overline{C}'''')$$

Now for C substitute $\bar{x}C_0+xC_1$ and expand about x to get $G = G_0\bar{x}+G_1x$ where

$$G_0 = ((B"'+B""+A_0)(\bar{B}"'+\bar{B}""+A_1)+C_0"'+C_0"")$$
$$((\bar{B}"'+\bar{B}""+A_0)(B"'+B""+A_1)+\bar{C}_0"'+\bar{C}_0"")$$
$$G_1 = ((B"'+B""+A_0)(\bar{B}"'+\bar{B}""+A_1)+C_1"'+C_1"")$$
$$((\bar{B}"'+\bar{B}""+A_0)(B"'+B""+A_1)+\bar{C}_1"'+\bar{C}_1"")$$

similarly

$$\bar{G}_0 = ((B"'+B""+\bar{A}_0)(\bar{B}"'+\bar{B}""+\bar{A}_1)+C_0"'+C_0"")$$
$$((\bar{B}"'+\bar{B}""+\bar{A}_0)(B"'+B""+\bar{A}_1)+\bar{C}_0"'+\bar{C}_0"")$$
$$\bar{G}_1 = ((B"'+B""+\bar{A}_0)(\bar{B}"'+\bar{B}""+\bar{A}_1)+C_1"'+C_1"")$$
$$((\bar{B}"'+\bar{B}""+\bar{A}_0)(B"'+B""+\bar{A}_1)+\bar{C}_1"'+\bar{C}_1"")$$

This portion of the network consists of 24 gates and uses 64 gate inputs.

By P6 the A_0 type terms are of depth

$$t_6(A) \le (-1/\log \alpha) \log(\alpha^5 n) + 1 + S$$
$$=(-1/\log \alpha) \log n + 1 + S -5$$

Thus, the depth bound for A_0 type terms is 5 less than the hypothesized depth bound for G_0. But the latest time at which A_0 type terms in the expression must be available is T-4, so there is no depth contradiction.

By P4 the B"' type terms are of depth

$$t_4(B) \le (-1/\log\alpha) \log(\alpha^2 n) - 1 + S$$
$$=(-1/\log\alpha) \log n + 1 + S -4$$

Thus, the depth bound for B"' type terms is 4 less than

the hypothesized depth bound for G_0. But the latest time at which B"' type terms in the expression must be available is T-4, so there is no depth contradiction.

By P8 the C_0''' type terms are of depth

$$t_8(C) \leq (-1/\log \alpha) \log(\alpha n) + S$$
$$=(-1/\log \alpha) \log n + 1 + S - 2$$

Thus, the depth bound for C_0''' type terms is 2 less than the hypothesized depth bound for G_0. But the latest time at which C_0''' type terms in the expression must be available is T-2, so there is no depth contradiction.

All variables used in the step 2 EXCLUSIVE-OR decomposition fall into one of these three categories, so there is no depth contradiction in the step 2 EXCLUSIVE-OR decomposition.

then for θ = AND

$$\overline{G} = \overline{(BC)}\,\overline{A}_0+BC\overline{A}_1$$
$$= (\overline{B}+\overline{C})\overline{A}_0+BC\overline{A}_1$$
$$= \overline{A}_0\overline{C}+(\overline{B}\overline{A}_0+B\overline{A}_1)C$$
$$G = (A_0+C)((B+A_0)(\overline{B}+A_1)+\overline{C})$$
$$= (A_0+C"'+C"")((B"'+B"'+A_0)(\overline{B}"'+\overline{B}"'+A_1)+\overline{C}"'+\overline{C}"")$$

Now for C substitute $\overline{x}C_0+xC_1$ and expand about x to get $G = G_0\overline{x}+G_1x$ where

$$G_0 = (A_0+C_0'''+C_0'''')$$
$$((B'''+B'''+A_0)(\bar{B}'''+\bar{B}'''+A_1)+\bar{C}_0'''+\bar{C}_0'''')$$
$$G_1 = (A_0+C_1'''+C_1'''')$$
$$((B'''+B'''+A_0)(\bar{B}'''+\bar{B}'''+A_1)+\bar{C}_1'''+\bar{C}_1'''')$$

similarly

$$\bar{G}_0 = (\bar{A}_0+C_0'''+C_0'''')$$
$$((B'''+B'''+\bar{A}_0)(\bar{B}'''+\bar{B}'''+\bar{A}_1)+\bar{C}_0'''+\bar{C}_0'''')$$
$$\bar{G}_1 = (\bar{A}_0+C_1'''+C_1'''')$$
$$((B'''+B'''+\bar{A}_0)(\bar{B}'''+\bar{B}'''+\bar{A}_1)+\bar{C}_1'''+\bar{C}_1'''')$$

All terms are satisfactory, based upon the previous arguments. Thus there is no depth contradiction in the step 2 AND decomposition.

then for θ = OR

$$\bar{G} = \overline{(B+C)}\bar{A}_0+(B+C)\bar{A}_1$$
$$= \bar{B}\bar{C}\bar{A}_0+(B+C)\bar{A}_1$$
$$= (\bar{B}\bar{A}_0+B\bar{A}_1)\bar{C}+\bar{A}_1 C$$
$$G = ((B+A_0)(\bar{B}+A_1)+C)(A_1+\bar{C})$$
$$= ((B'''+B''''+A_0)(\bar{B}'''+\bar{B}''''+A_1)+C'''+C'''')(A_1+\bar{C}'''+\bar{C}'''')$$

Now for C substitute $\bar{x}C_0+xC_1$ and expand about x to get $G = G_0\bar{x}+G_1x$ where

$$G_0 = ((B'''+B''''+A_0)(\bar{B}'''+\bar{B}''''+A_1)+C_0'''+C_0'''')$$
$$(A_1+\bar{C}_0'''+\bar{C}_0'''')$$
$$G_1 = ((B'''+B''''+A_0)(\bar{B}'''+\bar{B}''''+A_1)+C_1'''+C_1'''')$$
$$(A_1+\bar{C}_1'''+\bar{C}_1'''')$$

similarly

$$\overline{G}_0 = ((B''' + B'''' + \overline{A}_0)(\overline{B}''' + \overline{B}'''' + \overline{A}_1) + C_0''' + C_0'''')$$
$$(\overline{A}_1 + \overline{C}_0''' + \overline{C}_0'''')$$
$$\overline{G}_1 = ((B''' + B'''' + \overline{A}_0)(\overline{B}''' + \overline{B}'''' + \overline{A}_1) + C_1''' + C_1'''')$$
$$(\overline{A}_1 + \overline{C}_1''' + \overline{C}_1'''')$$

All term types used have been evaluated in other parts of step 2 and been found satisfactory, so there is no depth contradiction in step 2.

STEP 3: If $|B| > \alpha^2 n$

Then $|C| < n - \alpha^2 n = (1-\alpha^2)n = \alpha^3 n$

since $\alpha^3 + \alpha^2 - 1 = 0$

then for θ = EXCLUSIVE-OR

$$\overline{G} = (BC + \overline{B}\overline{C})\overline{A}_0 + (B\overline{C} + \overline{B}C)\overline{A}_1$$
$$= (\overline{C}\overline{A}_0 + C\overline{A}_1)\overline{B} + (C\overline{A}_0 + \overline{C}\overline{A}_1)B$$
$$G = ((C + A_0)(\overline{C} + A_1) + B)((\overline{C} + A_0)(C + A_1) + \overline{B})$$

By using arguments similar to those in step 2 for EXCLUSIVE-OR, the following constructions are obtained:

$$G_0 = ((C_0''' + C_0'''' + A_0)(\overline{C}_0''' + \overline{C}_0'''' + A_1) + B''' + B'''')$$
$$((\overline{C}_0''' + \overline{C}_0'''' + A_0)(C_0''' + C_0'''' + A_1) + \overline{B}''' + \overline{B}'''')$$
$$G_1 = ((C_1''' + C_1'''' + A_0)(\overline{C}_1''' + \overline{C}_1'''' + A_1) + B''' + B'''')$$
$$((\overline{C}_1''' + \overline{C}_1'''' + A_0)(C_1''' + C_1'''' + A_1) + \overline{B}''' + \overline{B}'''')$$
$$\overline{G}_0 = ((C_0''' + C_0'''' + \overline{A}_0)(\overline{C}_0''' + \overline{C}_0'''' + \overline{A}_1) + B''' + B'''')$$
$$((\overline{C}_0''' + \overline{C}_0'''' + \overline{A}_0)(C_0''' + C_0'''' + \overline{A}_1) + \overline{B}''' + \overline{B}'''')$$
$$\overline{G}_1 = ((C_1''' + C_1'''' + \overline{A}_0)(\overline{C}_1''' + \overline{C}_1'''' + \overline{A}_1) + B''' + B'''')$$

$$((\bar{C}_1''' + \bar{C}_1'''' + \bar{A}_0)(C_1''' + C_1'''' + \bar{A}_1) + \bar{B}''' + \bar{B}''''))$$

This portion of the network consists of 36 gates and uses 96 gate inputs.

The argument for A_0 type terms is identical to the argument for A_0 type terms in step 2, so there is no depth contradiction with the A_0 type terms.

By P4 the B''' type terms are of depth

$$t_4(B) \leq (-1/\log \alpha) \log(n) - 1 + S$$
$$= (-1/\log \alpha) \log n + 1 + S - 2$$

Thus, the depth bound for B''' type terms is 2 less than the hypothesized depth bound for G_0. But the latest time at which B''' type terms in the expression must be available is T-2, so there is no depth contradiction.

By P8 the C_0''' type terms are of depth

$$t_8(C) \leq (-1/\log \alpha) \log(\alpha^3 n) + S$$
$$= (-1/\log \alpha) \log n + 1 + S - 4$$

Thus, the depth bound for C_0''' type terms is 4 less than the hypothesized depth bound for G_0. But the latest time at which C_0''' type terms in the expression must be available is T-4, so there is no depth contradiction.

All variables used in the step 3 EXCLUSIVE-OR decomposition fall into one of these three categories, so there is no depth contradiction in the step 3 EXCLUSIVE-OR decomposition.

then for θ = AND

$$\begin{aligned}
\bar{G} &= \overline{(BC)}\,\bar{A}_0+BC\bar{A}_1 \\
&= (\bar{B}+\bar{C})\,\bar{A}_0+BC\bar{A}_1 \\
&= \bar{A}_0\bar{B}+(\bar{C}\bar{A}_0+C\bar{A}_1)\,B \\
G &= (A_0+B)\,((C+A_0)\,(\bar{C}+A_1)+\bar{B})
\end{aligned}$$

By using arguments similar to those in step 2 for AND, the following constructions are obtained:

$$\begin{aligned}
G_0 &= (A_0+B^{\prime\prime\prime}+B^{\prime\prime\prime\prime}) \\
&\qquad ((C_0^{\prime\prime\prime}+C_0^{\prime\prime\prime\prime}+A_0)\,(\bar{C}_0^{\prime\prime\prime}+\bar{C}_0^{\prime\prime\prime\prime}+A_1)+\bar{B}^{\prime\prime\prime}+\bar{B}^{\prime\prime\prime\prime}) \\
G_1 &= (A_0+B^{\prime\prime\prime}+B^{\prime\prime\prime\prime}) \\
&\qquad ((C_1^{\prime\prime\prime}+C_1^{\prime\prime\prime\prime}+A_0)\,(\bar{C}_1^{\prime\prime\prime}+\bar{C}_1^{\prime\prime\prime\prime}+A_1)+\bar{B}^{\prime\prime\prime}+\bar{B}^{\prime\prime\prime\prime}) \\
\bar{G}_0 &= (\bar{A}_0+B^{\prime\prime\prime}+B^{\prime\prime\prime\prime}) \\
&\qquad ((C_0^{\prime\prime\prime}+C_0^{\prime\prime\prime\prime}+\bar{A}_0)\,(\bar{C}_0^{\prime\prime\prime}+\bar{C}_0^{\prime\prime\prime\prime}+\bar{A}_1)+\bar{B}^{\prime\prime\prime}+\bar{B}^{\prime\prime\prime\prime}) \\
\bar{G}_1 &= (\bar{A}_0+B^{\prime\prime\prime}+B^{\prime\prime\prime\prime}) \\
&\qquad ((C_1^{\prime\prime\prime}+C_1^{\prime\prime\prime\prime}+\bar{A}_0)\,(\bar{C}_1^{\prime\prime\prime}+\bar{C}_1^{\prime\prime\prime\prime}+\bar{A}_1)+\bar{B}^{\prime\prime\prime}+\bar{B}^{\prime\prime\prime\prime})
\end{aligned}$$

All terms are satisfactory, based upon the previous arguments.

then for θ = OR

$$\bar{G} = \overline{(B+C)}\,\bar{A}_0+(B+C)\,\bar{A}_1$$

$$= \bar{B}\bar{C}\bar{A}_0+(B+C)\bar{A}_1$$

$$= (\bar{C}\bar{A}_0+C\bar{A}_1)\bar{B}+\bar{A}_1B$$

$$G = ((C+A_0)(\bar{C}+A_1)+B)(A_1+\bar{B})$$

By using arguments similar to those in step 2 for OR, the following constructions are obtained:

$$G_0 = ((C_0^{\prime\prime\prime}+C_0^{\prime\prime\prime\prime}+A_0)(\bar{C}_0^{\prime\prime\prime}+\bar{C}_0^{\prime\prime\prime\prime}+A_1)+B^{\prime\prime\prime}+B^{\prime\prime\prime\prime})(A_1+\bar{B}^{\prime\prime\prime}+\bar{B}^{\prime\prime\prime\prime})$$

$$G_1 = ((C_1^{\prime\prime\prime}+C_1^{\prime\prime\prime\prime}+A_0)(\bar{C}_1^{\prime\prime\prime}+\bar{C}_1^{\prime\prime\prime\prime}+A_1)+B^{\prime\prime\prime}+B^{\prime\prime\prime\prime})(A_1+\bar{B}^{\prime\prime\prime}+\bar{B}^{\prime\prime\prime\prime})$$

$$\bar{G}_0 = ((C_0^{\prime\prime\prime}+C_0^{\prime\prime\prime\prime}+\bar{A}_0)(\bar{C}_0^{\prime\prime\prime}+\bar{C}_0^{\prime\prime\prime\prime}+\bar{A}_1)+B^{\prime\prime\prime}+B^{\prime\prime\prime\prime})(\bar{A}_1+\bar{B}^{\prime\prime\prime}+\bar{B}^{\prime\prime\prime\prime})$$

$$\bar{G}_1 = ((C_1^{\prime\prime\prime}+C_1^{\prime\prime\prime\prime}+\bar{A}_0)(\bar{C}_1^{\prime\prime\prime}+\bar{C}_1^{\prime\prime\prime\prime}+\bar{A}_1)+B^{\prime\prime\prime}+B^{\prime\prime\prime\prime})(\bar{A}_1+\bar{B}^{\prime\prime\prime}+\bar{B}^{\prime\prime\prime\prime})$$

All term types used have been evaluated in other parts of step 3 and been found satisfactory, so there is no depth contradiction in step 3. Thus, there is no depth contradiction in the P6 decomposition.

The combination equipment worst case is 36 gates.

$$\begin{aligned} \text{GATES}(E) &\le \text{GATES}(A)+\text{GATES}(B)+\text{GATES}(C)+36 \\ &\le K|A|-K+K|B|-K+K|C|-K+36 \\ &= K(|A|+|B|+|C|)-3K+36 \\ &= K|E|-K-2K+36 \end{aligned}$$

thus for $K = 18$,

$$\text{GATES}(E) \le K|E|-K$$

Thus there is no contradiction in the P6 equipment hypothesis.

The extension of P7 is provided by the following argument:

The arguments and construction for P7 are essentially the same as for P6, except that the two terms at the output of the P6 constructions are separated and treated independently as G' and G" etc.

The extension of P8 is provided by the following argument:

The arguments and construction for P8 are essentially the same as for P5, except that the two terms at the output of the P5 constructions are separated and treated independently as G"' and G"" etc.

Thus the proof for the 3-input construction is complete.

The implication graph for this construction is sufficiently complex that it is best represented in the form of an adjacency matrix, as in figure 20. (Each X corresponds to a line segment directed from the node

corresponding to the row to the node corresponding to the column in which the X is located.)

	P1	P2	P3	P4	P5	P6	P7	P8
P1	X	X	X		X		X	
P2	X	X		X		X		X
P3	X	X		X		X		X
P4	X	X	X		X		X	
P5			X		X		X	
P6				X		X		X
P7				X		X		X
P8			X		X		X	

FIGURE 20 3-INPUT LINEAR EQUIPMENT IMPLICATION MATRIX

4.13 B2 TO 4-INPUT NAND

The construction used to provide linear equipment growth for B2 to 4-input NAND conversion provides an upper bound on network depth of $2 \log |E| + O(1)$. The total equipment required is upper bounded by $K\,|E| - K$, where K is no greater than 12. The number of gate inputs used varies with the Boolean operators. For EXCLUSIVE-OR all gate inputs are used, however for AND and OR operators at separations, there are unused gate inputs. The maximum gate

fan-out used is 4 (not including input literal fan-out).

With this particular construction, the results will be represented in terms of 4-input AND and OR gates which will alternate from the OR gate at the root, thus leaving a simple conversion to NAND gates for the end.

ASSERTION:

Any Boolean formula over the basis B2 can be restructured so that it describes a network of NAND gates with maximum fan-in of 4, with $O(|E|)$ equipment requirements, and with depth bounded from above by $2 \log |E| + O(1)$.

Proof:

Let E be a Boolean expression with $|E| \geq 1$.

Define $t_1(E)$ to be the minimum depth of any network realizing both expression E and $\bar{E}$ with total equipment $\leq K |E| - K$.

Define $t_2(E)$ to be the minimum depth of any network with outputs E''', E'''', $\bar{E}'''$, and $\bar{E}''''$ with total equipment $\leq K |E| - K$, where

$$E = E''' + E'''' \quad \text{and} \quad \bar{E} = \bar{E}''' + \bar{E}''''.$$

Define G to be a Boolean expression with a free

variable x such that the expression can be restructured as $\bar{x}G_0+xG_1$. Let $t_3(G)$ be the minimum depth of any network which simultaneously realizes G_0', G_0'', G_1', G_1'', $\bar{G}_0'$, $\bar{G}_0''$, $\bar{G}_1'$, and $\bar{G}_1''$ with total equipment $\leq K\,|E| - K$, where

$$G_0 = G_0'G_0'' \quad , \quad \bar{G}_0 = \bar{G}_0'\bar{G}_0'',$$

$$G_1 = G_1'G_1'', \text{ and } \quad \bar{G}_1 = \bar{G}_1'\bar{G}_1''.$$

Assume inductively:

P1: $t_1(E) \leq 2 \log |E| + S$

P2: $t_2(E) \leq 2 \log |E| - 1 + S$

P3: $t_3(G) \leq 2 \log |G| + S$

Clearly, S can be chosen large enough to satisfy the Basis step for the induction.

The extension of P1 is provided by the following algorithm, where $n = |E|$ and $n > 2$. For $n \leq 2$, the construction is obvious and GATES(E) $\leq K|E|-K$ for $K = 12$.

STEP 1: Using Lemma 1, decompose E as

$A \circ (B\theta C)$ with

$|B| = |C| < 1/2\ n$

$|A| = |E| - |B| - |C| \leq n - 1/2\ n = 1/2\ n$

If θ is OR

$$E = \overline{(B+C)}A_0+(B+C)A_1$$

$$= \bar{B}\bar{C}A_0+BA_1+CA_1$$

$= \bar{B}\bar{C}A_0'A_0''+BA_1'A_1''+CA_1'A_1''$

similarly

$\bar{E} = \bar{B}\bar{C}\bar{A}_0'\bar{A}_0''+B\bar{A}_1'\bar{A}_1''+C\bar{A}_1'\bar{A}_1''$

This portion of the network consists of 8 gates.

If θ is AND

$E = \overline{(BC)}A_0+BCA_1$

$= \bar{B}A_0+\bar{C}A_0+BCA_1$

$= \bar{B}A_0'A_0''+\bar{C}A_0'A_0''+BCA_1'A_1''$

$\bar{E} = \bar{B}\bar{A}_0'\bar{A}_0''+\bar{C}\bar{A}_0'\bar{A}_0''+BC\bar{A}_1'\bar{A}_1''$

This portion of the network consists of 8 gates.

If θ is EXCLUSIVE-OR

$E = (BC+\bar{B}\bar{C})A_0+(B\bar{C}+\bar{B}C)A_1$

$= BCA_0+\bar{B}\bar{C}A_0+B\bar{C}A_1+\bar{B}CA_1$

$= BCA_0'A_0''+\bar{B}\bar{C}A_0'A_0''+B\bar{C}A_1'A_1''+\bar{B}CA_1'A_1''$

$\bar{E} = BC\bar{A}_0'\bar{A}_0''+\bar{B}\bar{C}\bar{A}_0'\bar{A}_0''+B\bar{C}\bar{A}_1'\bar{A}_1''+\bar{B}C\bar{A}_1'\bar{A}_1''$

This portion of the network consists of 10 gates.

Thus in the worst case, B, C, and A_0' type terms are required in the construction by time T-2.

By P3 the A_0' type terms are of depth

$t_3(A) \leq 2\log(1/2\ n) + S$

$= 2\log n + S - 2$

Thus, the depth bound for A_0' type terms is 2 less than

the hypothesized depth bound for E. But the latest time at which A_0' type terms in the expression must be available is T-2, so there is no depth contradiction.

By P1 the B type terms are of depth

$$t_1(B) \le 2 \log(1/2\ n) + S$$
$$= 2 \log n + S - 2$$

Thus, the depth bound for B type terms is 2 less than the hypothesized depth bound for E. But the latest time at which B type terms in the expression must be available is T-2, so there is no depth contradiction.

The C type terms are similar to the B type terms and the same arguments apply, so there is no depth contradiction with the C type terms.

All variables used in the P1 decomposition fall into one of these three categories, so there is no depth contradiction in the P1 decomposition.

The combination equipment worst case is 10 gates.

$$\text{GATES}(E) \le \text{GATES}(A)+\text{GATES}(B)+\text{GATES}(C)+10$$
$$\le K|A|-K+K|B|-K+K|C|-K+10$$
$$= K(|A|+|B|+|C|)-3K+10$$
$$= K|E|-K-2K+10$$

thus for $K = 12$,

$$\text{GATES}(E) \le K|E|-K$$

Thus there is no contradiction in the P1 equipment hypothesis.

The extension of P2 is provided by the following algorithm, where $n = |E|$ and $n > 2$. For $n \leq 2$, the construction is obvious and GATES(E) $\leq K|E|-K$ for $K = 12$.

STEP 1: Using Lemma 1, decompose E as

$A \circ (B \theta C)$ with

$|B| \leq |C| < 1/2\ n$

$|A| = |E| - |B| - |C| \leq n - 1/2\ n = 1/2\ n$

If θ is OR

$$E = \overline{(B+C)}A_0 + (B+C)A_1$$

$$= \bar{B}\bar{C}A_0 + (B+C)A_1$$

$$E''' = (\bar{B}''' + \bar{B}'''')(\bar{C}''' + \bar{C}'''')A_0'A_0''$$

$$E'''' = (B''' + B'''' + C''' + C'''')A_1'A_1''$$

similarly

$$\bar{E}''' = (\bar{B}''' + \bar{B}'''')(\bar{C}''' + \bar{C}'''')\bar{A}_0'\bar{A}_0''$$

$$\bar{E}'''' = (B''' + B'''' + C''' + C'''')\bar{A}_1'\bar{A}_1''$$

This portion of the network consists of 7 gates.

If θ is AND

$$E = \overline{(BC)}A_0 + BCA_1$$

$$= (\bar{B} + \bar{C})A_0 + BCA_1$$

$$E''' = (\bar{B}''' + \bar{B}'''' + \bar{C}''' + \bar{C}'''')A_0'A_0''$$

$$E'''' = (B''' + B'''')(C''' + C'''')A_1'A_1''$$

$$\bar{E}''' = (\bar{B}'''+\bar{B}''''+\bar{C}'''+\bar{C}'''')\bar{A}_0'\bar{A}_0''$$

$$\bar{E}'''' = (B'''+B'''')(C'''+C'''')\bar{A}_1'\bar{A}_1''$$

This portion of the network consists of 7 gates.

If θ is EXCLUSIVE-OR

$$E = (B+\bar{C})(\bar{B}+C)A_0+(B+C)(\bar{B}+\bar{C})A_1$$

$$E''' = (B'''+B''''+\bar{C}'''+\bar{C}'''')(\bar{B}'''+\bar{B}''''+C'''+C'''')A_0'A_0''$$

$$E'''' = (B'''+B''''+C'''+C'''')(\bar{B}'''+\bar{B}''''+\bar{C}'''+\bar{C}'''')A_1'A_1''$$

$$\bar{E}''' = (B'''+B''''+\bar{C}'''+\bar{C}'''')(\bar{B}'''+\bar{B}''''+C'''+C'''')\bar{A}_0'\bar{A}_0''$$

$$\bar{E}'''' = (B'''+B''''+C'''+C'''')(\bar{B}'''+\bar{B}''''+\bar{C}'''+\bar{C}'''')\bar{A}_1'\bar{A}_1''$$

This portion of the network consists of 8 gates.

Thus in the worst case, B"' and C"' type terms are required in the construction by time T-2, while A_0' type terms are required by time T-1.

By P3 the A_0' type terms are of depth

$$t_3(A) \leq 2\log(1/2\ n) + S$$
$$= 2\log n - 1 + S - 1$$

Thus, the depth bound for A_0' type terms is 1 less than the hypothesized depth bound for E"'. But the latest time at which A_0' type terms in the expression must be available is T-1, so there is no depth contradiction.

By P2 the B"' type terms are of depth

$$t_2(B) \leq 2\log(1/2\ n) - 1 + S$$

$$= 2 \log n - 1 + S - 2$$

Thus, the depth bound for B"' type terms is 2 less than the hypothesized depth bound for E"'. But the latest time at which B"' type terms in the expression must be available is T-2, so there is no depth contradiction.

The C"' type terms are similar to the B"' type terms and the same arguments apply, so there is no depth contradiction with the C"' type terms.

All variables used in the P2 decomposition fall into one of these three categories, so there is no depth contradiction in the P2 decomposition.

The combination equipment worst case is 8 gates.

$$\begin{aligned} \mathrm{GATES(E)} &\le \mathrm{GATES(A)}+\mathrm{GATES(B)}+\mathrm{GATES(C)}+8 \\ &\le K|A|-K+K|B|-K+K|C|-K+8 \\ &= K(|A|+|B|+|C|)-3K+8 \\ &= K|E|-K-2K+8 \end{aligned}$$

thus for $K = 12$,

$$\mathrm{GATES(E)} \le K|E|-K$$

Thus there is no contradiction in the P2 equipment hypothesis.

The extension of P3 is provided by the following algorithm, where $n = |G|$ and $n > 2$. For $n \leq 2$, the construction is obvious and $\mathrm{GATES}(G) \leq K|G|-K$ for $K = 12$.

STEP 1: Using Lemma 2, decompose G as

$A \circ (B \theta C)$ with $|C| < 1/2\, n$,

C containing the free variable, x, and

$|A| = |E| - |B\theta C| \leq n - 1/2\, n = 1/2\, n.$

If θ is OR

$$\bar{G} = \overline{(B+C)}\bar{A}_0 + (B+C)\bar{A}_1$$
$$= \bar{B}\bar{C}\bar{A}_0 + (B+C)\bar{A}_1$$
$$= (\bar{C}\bar{A}_0 + C\bar{A}_1)\bar{B} + \bar{A}_1 B$$
$$G = ((C+A_0)(\bar{C}+A_1)+B)(A_1+\bar{B})$$
$$G' = (C+A_0)(\bar{C}+A_1)+B$$
$$= \bar{C}A_0 + CA_1 + B$$
$$G'' = A_1 + \bar{B}$$

Now for C substitute $\bar{x}C_0 + xC_1$ and expand about x to get

$G = G_0'G_0''\bar{x} + G_1'G_1''x$ where

$$G_0' = \bar{C}_0A_0 + C_0A_1 + B$$
$$G_0'' = A_1 + \bar{B}$$
$$G_1' = \bar{C}_1A_0 + C_1A_1 + B$$
$$G_1'' = A_1 + \bar{B}$$

giving finally

$$G_0' = \bar{C}_0'\bar{C}_0''A_0'A_0'' + C_0'C_0''A_1'A_1'' + B''' + B''''$$
$$G_0'' = A_1'A_1'' + \bar{B}''' + \bar{B}''''$$
$$G_1' = \bar{C}_1'\bar{C}_1''A_0'A_0'' + C_1'C_1''A_1'A_1'' + B''' + B''''$$

$G_1'' = A_1'A_1''+\bar{B}'''+\bar{B}''''$

$\bar{G}_0' = \bar{C}_0'\bar{C}_0''\bar{A}_0'\bar{A}_0''+C_0'C_0''\bar{A}_1'\bar{A}_1''+B'''+B''''$

$\bar{G}_0'' = \bar{A}_1'\bar{A}_1''+\bar{B}'''+\bar{B}''''$

$\bar{G}_1' = \bar{C}_1'\bar{C}_1''\bar{A}_0'\bar{A}_0''+C_1'C_1''\bar{A}_1'\bar{A}_1''+B'''+B''''$

$G_1'' = \bar{A}_1'\bar{A}_1''+\bar{B}'''+\bar{B}''''$

This portion of the network consists of 16 gates.

If θ is AND

$\bar{G} = \overline{(BC)}\bar{A}_0+BC\bar{A}_1$

$= (\bar{B}+\bar{C})\bar{A}_0+BC\bar{A}_1$

$= \bar{A}_0\bar{B}+(\bar{C}\bar{A}_0+C\bar{A}_1)B$

$G = (A_0+B)((C+A_0)(\bar{C}+A_1)+\bar{B})$

$G' = A_0+B$

$G'' = (C+A_0)(\bar{C}+A_1)+\bar{B}$

$= \bar{C}A_0+CA_1+\bar{B}$

Now for C substitute $\bar{x}C_0+xC_1$ and expand about x to get

$G = G_0'G_0''\bar{x}+G_1'G_1''x$ where

$G_0' = A_0+B$

$G_0'' = \bar{C}_0A_0+C_0A_1+\bar{B}$

$G_1' = A_0+B$

$G_1'' = \bar{C}_1A_0+C_1A_1+\bar{B}$

giving finally

$G_0' = A_0'A_0''+B'''+B''''$

$G_0'' = \bar{C}_0'\bar{C}_0''A_0'A_0''+C_0'C_0''A_1'A_1''+\bar{B}'''+\bar{B}''''$

$G_1' = A_0'A_0''+B'''+B''''$

$G_1'' = \bar{C}_1'\bar{C}_1''A_0'A_0''+C_1'C_1''A_1'A_1''+\bar{B}'''+\bar{B}''''$

$$\overline{G}'_0 = \overline{A}'_0\overline{A}''_0+B'''+B''''$$

$$\overline{G}''_0 = \overline{C}'_0\overline{C}''_0\overline{A}'_0\overline{A}''_0+C'_0C''_0\overline{A}'_1\overline{A}''_1+\overline{B}'''+\overline{B}''''$$

$$\overline{G}'_1 = \overline{A}'_0\overline{A}''_0+B'''+B''''$$

$$\overline{G}''_1 = \overline{C}'_1\overline{C}''_1\overline{A}'_0\overline{A}''_0+C'_1C''_1\overline{A}'_1\overline{A}''_1+\overline{B}'''+\overline{B}''''$$

This portion of the network consists of 16 gates.

If θ is EXCLUSIVE-OR

$$\overline{G} = (BC+\overline{B}\overline{C})\overline{A}_0+(B\overline{C}+\overline{B}C)\overline{A}_1$$

$$= (\overline{C}\overline{A}_0+C\overline{A}_1)\overline{B}+(C\overline{A}_0+\overline{C}\overline{A}_1)B$$

$$G = ((C+A_0)(\overline{C}+A_1)+B)((\overline{C}+A_0)(C+A_1)+\overline{B})$$

$$= (\overline{C}A_0+CA_1+B)(CA_0+\overline{C}A_1+\overline{B})$$

$$G' = \overline{C}A_0+CA_1+B$$

$$G'' = CA_0+\overline{C}A_1+\overline{B}$$

Now for C substitute $\overline{x}C_0+xC_1$ and expand about x to get

$G = G'_0G''_0\overline{x}+G'_1G''_1x$ where

$$G'_0 = \overline{C}_0A_0+C_0A_1+B$$

$$G''_0 = C_0A_0+\overline{C}_0A_1+\overline{B}$$

$$G'_1 = \overline{C}_1A_0+C_1A_1+B$$

$$G''_1 = C_1A_0+\overline{C}_1A_1+\overline{B}$$

giving finally

$$G'_0 = \overline{C}'_0\overline{C}''_0A'_0A''_0+C'_0C''_0A'_1A''_1+B'''+B''''$$

$$G''_0 = C'_0C''_0A'_0A''_0+\overline{C}'_0\overline{C}''_0A'_1A''_1+\overline{B}'''+\overline{B}''''$$

$$G'_1 = \overline{C}'_1\overline{C}''_1A'_0A''_0+C'_1C''_1A'_1A''_1+B'''+B''''$$

$$G''_1 = C'_1C''_1A'_0A''_0+\overline{C}'_1\overline{C}''_1A'_1A''_1+\overline{B}'''+\overline{B}''''$$

$$\overline{G}'_0 = \overline{C}'_0\overline{C}''_0\overline{A}'_0\overline{A}''_0+C'_0C''_0\overline{A}'_1\overline{A}''_1+B'''+B''''$$

$$\overline{G}''_0 = C'_0C''_0\overline{A}'_0\overline{A}''_0+\overline{C}'_0\overline{C}''_0\overline{A}'_1\overline{A}''_1+\overline{B}'''+\overline{B}''''$$

$$\bar{G}_1' = \bar{C}_1'\bar{C}_1''\bar{A}_0'\bar{A}_0''+C_1'C_1''\bar{A}_1'\bar{A}_1''+B'''+B''''$$

$$\bar{G}_1'' = C_1'C_1''\bar{A}_0'\bar{A}_0''+\bar{C}_1'\bar{C}_1''\bar{A}_1'\bar{A}_1''+\bar{B}'''+\bar{B}''''$$

This portion of the network consists of 24 gates.

Thus in the worst case in the constructions, A_0 type and C_0 type terms are required by time T-2, while B type terms are required by time T-1.

By P3 the A_0' type terms are of depth

$$t_3(A) \leq 2 \log(1/2\ n) + S$$
$$= 2 \log n + S - 2$$

Thus, the depth bound for A_0' type terms is 2 less than the hypothesized depth bound for G_0'. But the latest time at which A_0' type terms in the expression must be available is T-2, so there is no depth contradiction.

By P2 the $B''\,'$ type terms are of depth

$$t_2(B) \leq 2 \log(n) - 1 + S$$
$$= 2 \log n + S - 1$$

Thus, the depth bound for $B''\,'$ type terms is 1 less than the hypothesized depth bound for G_0'. But the latest time at which $B''\,'$ type terms in the expression must be available is T-1, so there is no depth contradiction.

The C_0 type terms are similar to the A_0 type terms and the same arguments apply, so there is no depth

contradiction with the C_0 type terms.

All variables used in the P3 decomposition fall into one of these three categories, so there is no depth contradiction in the P3 decomposition.

The combination equipment worst case is 24 gates.

$$\begin{aligned} GATES(E) &\le GATES(A)+GATES(B)+GATES(C)+24 \\ &\le K|A|-K+K|B|-K+K|C|-K+24 \\ &= K(|A|+|B|+|C|)-3K+24 \\ &= K|E|-K-2K+24 \end{aligned}$$

thus for $K = 12$,

$$GATES(E) \le K|E|-K$$

Thus there is no contradiction in the P3 equipment hypothesis.

Thus the entire proof is complete.

The implication graph for this construction is shown in figure 21.

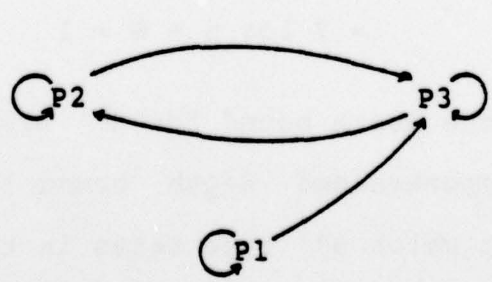

FIGURE 21 4-INPUT LINEAR EQUIPMENT IMPLICATION GRAPH

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VITA

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In 1970, while on active duty with the USAF, he earned the M.S. degree from the Air Force Institute of Technology. He went to the University of Illinois in 1976 to begin work on the Ph.D. degree.

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